GEODYN Systems Description Volume 1

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1 THE GEODYN II PROGRAM

The GEODYN II program, which evolved from GEODYN, is a totally redesigned software system. It has been written for the NASA/Goddard Space Flight Center by EG&G Washington Analytical Services Center, Inc., and has been operational since November 1985. The main user group is NASA/GSFC, where GEODYN II first operated on an IBM 3081 Computer and on a CDC Cyber 205 vector processing computer. The highly efficient optimized operation on the Cyber has been the main reason GEODYN has been redesigned. These computers are not available any more at GSFC, and GEODYN now operates on a Cray Y-MP. In the mid-1990.s, GEODYN was optimized to take advantage of the Cray parallel processors.

With the evolution of small workstations, GEODYN II has mainly been used on HP 900 series computers and during the last few years on SUN workstations. Now in the year 2002, the maintenance of the GEODYN code and executables is done entirely on a SUN workstation.

Currently, the maintenance and development of GEODYN II is performed by RAYTHEON ITSS under contract.

Like GEODYN, GEODYN II is used extensively for satellite orbit determination, geodetic parameter estimation, tracking instrument calibration, satellite orbit prediction, as well as for many other items relating to applied research in satellite geodesy using virtually all types of satellite tracking data.

The GEODYN II documentation is web-based and features an Operations Manual, a File Description chapter, and maintenance history.

2 THE ORBIT AND GEODETIC PARAMETER ESTIMATION PROBLEM

The purpose of this section is to provide an understanding of the relationship between the various elements in the solution of the orbit and geodetic parameter estimation problem. As such, it is a general statement of the problem and serves to coordinate the detailed solutions to each element in the problem presented in the sections that follow.

The problem is divided into two parts:

- the orbit prediction problem, and
- the parameter estimation problem.

The solution to the first of these problems corresponds to GEODYN's orbit generation mode. The solution to the latter corresponds to GEODYN's data reduction mode and of course is based on the solution to the former.

The reader should note that there are two key choices that dramatically affect the GEODYN solution structure:

- Cowell's method of integrating the orbit, and
- a Bayesian least squares statistical estimation procedure for the parameter estimation problem.

2.1 THE ORBIT PREDICTION PROBLEM

There are a number of approaches to orbit prediction. The GEODYN approach is to use Cowell's method, which is the direct numerical integration of the satellite equations of motion in rectangular coordinates. The initial conditions for these differential equations are the epoch position and velocity; the accelerations of the satellite must be evaluated.

The acceleration-producing forces that are currently modeled in GEODYN are the effects of:

- the geopotential,
- the luni-solar potentials,
- planetary potentials,
- radiation pressure,
- solid Earth and ocean tidal potential,
- atmospheric drag, and
- general acceleration.

Perhaps the most outstanding common feature of these forces is that they are functions of the position of the satellite relative to the Earth, Sun, Moon, or Planets and of the Sun and Moon relative to the Earth. Only atmospheric drag is a function of an additional quantity,[1] specifically, the relative velocity of the satellite with respect to the atmosphere.

The accurate evaluation of the acceleration of a satellite therefore involves the solution in two concomitant problems:

- The accurate modeling of each force on the satellite - Earth - Sun - Moon - Planet relationship, and

- The precise modeling of the motions of the Earth, Sun, Moon, and Planets.

The specific details for each model in these solutions are given elsewhere in Sections 3, 4, and 8. The question of how these models fit together is, in effect, the question of appropriate coordinate systems.

The key factor in the selection of coordinate systems for the satellite orbit prediction problem is the motion of the Earth. For the purposes of GEODYN, this motion consists of:

1. precession and nutation, and
2. rotation.

We are considering here the motion of the solid body of the Earth, as versus the slippage in the Earth's crust (polar motion) which just affects the position of the observer.

The precession and nutation define the variation in:

- The direction of the spin axis of the Earth (+ Z), and
- The direction of the true equinox of date (+ X).

These directions define the (geocentric) true of date coordinate system.

The rotation rate of the Earth is the time rate of change of the Greenwich hour angle θ_g between the Greenwich meridian and the true equinox of date. Thus the Earth-fixed system differs from the true of date system according to the rotation angle θ_g.

The equations of motion for the satellite must be integrated in an inertial coordinate system. The GEODYN inertial system is defined as the true of date system corresponding to $0.^h0$ of a reference epoch.

The coordinate systems in which the accelerations due to each physical effect are evaluated should be noted. The geopotential effects are evaluated in the Earth-fixed system, and then transformed to true of date to be combined with the other effects. The others are evaluated in the true of date system. The total acceleration is then transformed to the reference inertial system for use in the integration procedure.

The integration procedure used in GEODYN is a predictor-corrector type with a fixed time step. There is an optional variable step procedure. As the integration algorithms used provide for output on an even step, an interpolation procedure is required.

2.2 THE PARAMETER ESTIMATION PROBLEM

Let us consider the relationships between the observations O_i, their corresponding computed values C_i, and $\vec{P}$ the vector of parameters to be determined. These relationships are given by:

$$O_i - C_i = -\sum_j \frac{\partial C_i}{\partial P_j} dP_j + dO_i \qquad (2.2 - 1)$$

where:

i denotes the ith observation or association with it

P_j is the correction to the j^{th} parameter, and

dO_i is the error of observation associated with the i^{th} observation.

The basic problem of parameter estimation is to determine a solution to these equations.

The role of data preprocessing is quite apparent from these equations. First, the observation and its computed equivalent must be in a common time and spatial reference system. Second, there are certain physical effects such as atmospheric refraction which do not significantly vary by any likely change in the parameters represented by $\bar{P}$.

These computations and corrections may equally well be applied to the observations or to their computed values. Furthermore, the relationship between the computed value and the model parameters $\bar{P}$ is, in general, nonlinear, and hence the computed values may have to be evaluated several times in the estimation procedure. Thus a considerable increase in computational efficiency may be attained by applying these computations and corrections to the observations; i.e., to preprocess the data.

The preprocessed observations used by GEODYN are directly related to the position and/or velocity of the satellite relative to the observer at the given observation time. These relationships are geometric, hence computed equivalents for these observations are obtained by applying these geometric relationships to the computed values for the positions and velocities of the satellite and the observer at the desired time.

Associated with each measurement from each observing station is a (known) statistical uncertainty. This uncertainty is a statistical property of the noise on the observations. It is the reason a statistical estimation procedure is required for the GEODYN parameter determination.

It should be noted that dO_i, the measurement error, is not the same as the noise on the observations. The dO_i account for all of the discrepancy (O_i - C_i) which is not accounted for by the corrections to the parameters $d\bar{P}$.

These dO_i represent both:

- The contribution from the noise on the observation, and
- The incompleteness of the mathematical model represented by the parameters $\bar{P}$.

By this last expression, we mean either that the parameter set being determined is insufficient to model the physical situation, or that the functional form of the model is inadequate.

GEODYN has two different ways of dealing with these errors of observation:

1. The measurement model includes both a constant bias and a timing bias that may be determined.
2. There is an automatic editing procedure to delete bad (statistically unlikely) measurements.

The nature of the parameters to be determined has a significant effect on the functional structure of the solution. In GEODYN, these parameters are:

- The position and velocity of the satellite at epoch. These are the initial conditions for the equations of motion.
- Force model parameters. These affect the motion of the satellite.
- Station positions and biases for station measurement types. These do not affect the motion of the satellite.

Thus, the parameters to be determined are implicitly partitioned into a set $\bar{\alpha}$, which is not concerned with the dynamics of the satellite motion and a set $\bar{\beta}$, which are.

The computed value C_i, for each observation O_i, is a function of $\bar{r}_{ob}$, the Earth-fixed position vector of the station, and $\bar{x}_t$ = [x,y,z,$\dot{x}$, $\dot{y}$, $\dot{z}$], the true of date position and velocity vector of the satellite at the desired observation time. When measurement biases are used, C_i is also a function of $\bar{B}$, the biases associated with the particular station measurement type.

Let us consider the effect of the given partitioning on the required partial derivatives in the observational equations. The $\frac{\partial C_i}{\partial P}$ becomes:

$$\frac{\partial C_i}{\partial \bar{\alpha}} = \left[\frac{\partial C_i}{\partial r_{ob}}, \frac{\partial C_i}{\partial \bar{B}}\right] \tag{2.2 - 2}$$

$$\frac{\partial C_i}{\partial \bar{\beta}} = \frac{\partial C_i}{\partial \bar{x}_t}\frac{\partial \bar{x}_t}{\partial \bar{\beta}} \tag{2.2 - 3}$$

The partial derivatives $\frac{\partial \bar{x}_t}{\partial \bar{\beta}}$ are called the variational partials. While the other partial derivatives on the right-hand side of the equations above are computed from the measurement model at the given time, the variational partials must be obtained by integrating the variational equations. As will be shown in Section 8, these equations are similar to the equations of motion.

The need for the above-mentioned variational partials obviously has a dramatic effect on any solution to the observational equations. In addition to integrating the equations of motion to generate an orbit, the solution requires that the variational equations be integrated.

We have heretofore discussed the elements of the observational equations; we shall now discuss the solution of these equations, i.e., the statistical estimation scheme.

There are a number of estimation schemes that can be used. The method used in GEODYN is a batch scheme that uses all observations simultaneously to estimate the parameter set. The alternative would be a sequential scheme that uses the observations sequentially to calculate an updated set of parameters from each additional observation. Although batch and sequential schemes are essentially equivalent, practical numerical problems often occur with sequential schemes, especially when processing highly accurate observations. Therefore, a batch scheme was chosen.

The particular method selected for GEODYN is a partitioned Bayesian least squares method as detailed in Section 10. A Bayesian method is selected because such a scheme utilizes meaningful a priori information. The partitioning is such that the only arrays that must be simultaneously in core are arrays associated with parameters common to all satellite arcs and arrays pertaining to the arc being processed. Its purpose is to dramatically reduce the core storage requirements of the program without any significant cost in computation time.

There is an interesting side related to the use of a priori information in practice. The use of a priori information for the parameters guarantees that the estimation procedure will mechanically operate (but not necessarily converge). The user must ensure that his data contains information relating to the parameters he wishes to be determined.

[1] Not to be confused with the .fixed. parameters in the models.

3 THE MOTION OF THE EARTH AND RELATED COORDINATE SYSTEMS

The major factor in satellite dynamics is the gravitational attraction of the Earth. Because of the (usual) closeness of the satellite and its primary, the Earth cannot be considered a point mass, and hence any model for the dynamics must contain at least an implicit mass distribution. The concern of this section is the motion of this mass distribution and its relation to coordinate systems.

We will first consider the meaning of this motion of the Earth in terms of the requisite coordinate systems for the orbit prediction problem.

The choice of appropriate coordinate systems is controlled by several factors:

- In the case of a satellite moving in the Earth's gravitational field, the most suitable reference system for orbit computation is a system with its origin at the Earth's center of mass, referred to as a geocentric reference system.
- The satellite equations of motion must be integrated in an inertial coordinate system.
- The Earth is rotating at a rate , which is the time rate of change of the Greenwich hour angle. This angle is the hour angle of the true equinox of date with respect to the Greenwich meridian as measured in the equatorial plane.
- The Earth both precesses and nutates, thus changing the directions of both the Earth's spin axis and the true equinox of date in inertial space.

The motions of the Earth referred to here are of course those of the "solid body" of the Earth, the motion of the primary mass distribution. The slippage of the Earth's crust is considered elsewhere in Section 5.4 (polar motion).

3.1 THE TRUE OF DATE COORDINATE SYSTEM

Let us consider that at any given time, the spin axis of the Earth (+ Z) and the direction of the true equinox of date (+ X) may be used to define a right-handed geocentric coordinate system. This system is known as the true of date coordinate system. The coordinate systems of GEODYN will be defined in terms of this system.

3.2 THE INERTIAL COORDINATE SYSTEM

The inertial coordinate system of GEODYN is the true of date coordinate system defined at 0h.0 of the reference epoch for each satellite. This is the system in which the satellite equations of motion are integrated.

This is a right-handed, Cartesian, geocentric coordinate system with the X axis directed along the true equinox of $0^h.0$ of the reference day and with the Z axis directed along the Earth's spin axis toward north at the same time. The Y axis is of course defined so that the coordinate system is orthogonal.

It should be noted that the inertial system differs from the true of date system for a time not equal to 0^h of the reference day by the variation in time of the directions of the Earth's spin axis and the true equinox of date. This variation is described by the effects of precession and nutation.

3.3 THE EARTH-FIXED COORDINATE SYSTEM

The Earth-fixed coordinate system is geocentric, with the Z axis pointing north along the axis of rotation and with the X axis in the equatorial plane pointing toward the Greenwich meridian. The system is orthogonal and right-handed; thus the Y axis is automatically defined. The Z axis in this system is aligned with the mean pole of the Earth, making earth-fixed coordinates a mean pole coordinate system.

This system is rotating with respect to the true of date coordinate system. The Z axis, the spin axis of the Earth, is common to both systems. The rotation rate is equal to the Earth's angular velocity. Consequently, the hour angle of the true equinox of date with respect to the Greenwich meridian (measured westward in the equatorial plane) is changing at a rate , equal to the angular velocity of the Earth.

3.4 TRANSFORMATION BETWEEN EARTH-FIXED AND TRUE OF DATE COORDINATES

The transformation between Earth-fixed and true of date coordinates is a simple rotation. The Z axis is common to both systems. The angle between X_i, the true of date X component vector, and X_e, the Earth-fixed component vector, is θ_g, the Greenwich hour angle. The Y component vectors are similarly related. These transformations for X_e, Y_e, X_i, Y_i, are:

$$
\begin{aligned}
X_e &= X_i \cos\theta_g + Y_i \sin\theta_g \\
Y_e &= -X_i \sin\theta_g + Y_i \cos\theta_g \\
X_i &= X_e \cos\theta_g - Y_e \sin\theta_g \\
Y_i &= X_e \sin\theta_g + Y_e \cos\theta_g
\end{aligned}
$$

The transformation of velocities requires taking into account the rotational velocity, , of the Earth-fixed system with respect to the true of date reference frame. The following relationships should be noted:

$$\frac{\partial X_e}{\partial \theta_g} = Y_e \frac{\partial Y_e}{\partial \theta_g} = -X_e \tag{3.4 - 1}$$

$$\frac{\partial X_i}{\partial \theta_g} = -Y_i \frac{\partial Y_i}{\partial \theta_g} = X_i \tag{3.4 - 2}$$

The velocity transformations are then:

$$\begin{aligned}
\dot{X}_e &= [\dot{X}_i \cos\theta_g + \dot{Y}_i \sin\theta_g] + Y_e \dot{\theta}_g \\
\dot{Y}_e &= [-\dot{X}_i \sin\theta_g + \dot{Y}_i \cos\theta_g] - X_e \dot{\theta}_g \\
\dot{X}_i &= [\dot{X}_e \cos\theta_g - \dot{Y}_e \sin\theta_g] - Y_i \dot{\theta}_g \\
\dot{Y}_i &= [\dot{X}_e \sin\theta_g + \dot{Y}_e \cos\theta_g] + X_i \dot{\theta}_g
\end{aligned}$$

The brackets denote the part of each transform that is a transformation identical to its coordinate equivalent.

These same transformations are used in the transformation of partial derivatives from the Earth-fixed system to the true of date system. For the k_{th} measurement, C_k, the partial derivative transformations are explicitly:

$$\frac{\partial C_k}{\partial X_i} = \left[\frac{\partial C_k}{\partial X_e} \cos\theta_g - \frac{\partial C_k}{\partial Y_e} \sin\theta_g\right] + \left[\frac{\partial C_k}{\partial \dot{X}_e} \sin\theta_g - \frac{\partial C_k}{\partial \dot{Y}_e} \cos\theta_g\right] \dot{\theta}_g \tag{3.4 - 3}$$

$$\frac{\partial C_k}{\partial Y_i} = \left[\frac{\partial C_k}{\partial X_e} \sin\theta_g + \frac{\partial C_k}{\partial Y_e} \cos\theta_g\right] + \left[\frac{\partial C_k}{\partial \dot{X}_e} \cos\theta_g - \frac{\partial C_k}{\partial \dot{Y}_e} \sin\theta_g\right] \dot{\theta}_g \tag{3.4 - 4}$$

$$\frac{\partial C_k}{\partial \dot{X}_i} = \left[\frac{\partial C_k}{\partial \dot{X}_e} \cos\theta_g - \frac{\partial C_k}{\partial \dot{Y}_e} \sin\theta_g\right] \tag{3.4 - 5}$$

$$\frac{\partial C_k}{\partial \dot{Y}_i} = \left[\frac{\partial C_k}{\partial \dot{X}_e} \sin\theta_g + \frac{\partial C_k}{\partial \dot{Y}_e} \cos\theta_g\right] \tag{3.4 - 6}$$

The brackets have the same meaning as before.

3.5 COMPUTATION OF THE GREENWICH HOUR ANGLE

The computation of the Greenwich hour angle is quite important because it provides the orientation of the Earth relative to the true of date system. The additional effects; i.e., to transform from true of date to true of reference, of precession and nutation (true of reference to inertial), are sufficiently small that early orbit analysis programs neglected them. Thus, this angle is the major variable in relating the Earth-fixed system to the inertial reference frame in which the satellite equations of motion are integrated.

The evaluation of θ_g is discussed in detail in the Explanatory Supplement [3-1]. When the DE-96 or DE-118 planetary ephemerides are used, θ_g is computed from the expression:

$$\theta_g = \theta_{g0} + \Delta t_2 \dot{\theta}_2 + \Delta\alpha \qquad (3.5 - 1)$$

where

- Δt_2 is the fractional UT part of a day for the time of interest,
- $\theta_g 0$ is the Greenwich hour angle on January 0.0 UT of the reference year,
- $\dot{\theta}_2$ is the mean daily rate of advance of Greenwich hour angle $(2\pi + \theta_1)$, and
- $\Delta\alpha$ is the equation of equinoxes (nutation in right ascension).

θ_{g0} for January 0.0 UT of the reference year is calculated using the following formula from the "Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac":

$\theta_{g0} = G.M.S.T. = 6^h 38^m 45.^s 836 + 8640184.^s 542 T_u + 0.^s 0929 T_u^2$

where

- G.M.S.T. is the Greenwich mean sidereal time of 0 hours UT.
- T_u is the number of Julian centuries of 36525 days of universal time elapsed since the epoch of Greenwich mean noon of 1900 January 0.
- The equation of the equinoxes, $\Delta\alpha$, is calculated from the values of the nutation angles interpolated from the JPL Chebyshev ephemeris tape.
- When the DE-200 planetary ephemeris is used, θ_{g0} is the Greenwich mean sidereal time 0^h UT of the reference date and is given in the USNO, CIRCULAR NO. 163.
- θ_{g0}= GMST = $24110^s.54841$ + $8640184^s.812866\mathrm{T}_u$ + $0^s.093104$ - $6^s.2 \times 10^- 6T_u^3$

3.6 PRECESSION AND NUTATION

The inertial coordinate system of GEODYN in which the equations of motion are integrated is defined by the true equator and equinox of date for $0.^h0$ of the reference day. However, the Earth-fixed coordinate system is related to the true equator and equinox of date at any given instant. Thus, it is necessary to consider the effects that change the orientation in space of the equatorial plane and the ecliptic plane.

These phenomena are:

- The combined gravitational effect of the moon and the sun on the Earth's equatorial bulge, and
- The effect of the gravitational pulls of the various planets on the Earth's orbit.

The first of these affects the orientation of the equatorial plane; the second affects the orientation of the ecliptic plane. Both affect the relationship between the inertial and Earth-fixed reference systems of GEODYN.

The effect of these phenomena is to cause precession and nutation, both for the spin axis of the Earth and for the ecliptic pole. This precession and nutation provides the relationship between the inertial system defined by the true equator and equinox of the reference date and the "instantaneous" inertial system defined by the true equator and equinox of date at any given instant. Let us consider the effect of each of these phenomena in greater detail.

The luni-solar effects cause the Earth's axis of rotation to precess and nutate about the ecliptic pole. This precession will not affect the angle between the equatorial plane and the ecliptic (the "obliquity of the ecliptic"), but will affect the position of the equinox in the ecliptic plane. Thus the effect of luni-solar precession is entirely in celestial longitude. The nutation will affect both; consequently, we have nutation in longitude and nutation in obliquity.

The effect of the planets on the Earth's orbit is both secular and periodic deviations. However, the ecliptic is defined to be the mean plane of the Earth's orbit. Periodic effects are not considered to be a change in the orientation of the ecliptic; they are considered to be a perturbation of the Earth's celestial latitude [3-1]. The secular effect of the planets on the ecliptic plane is separated into two parts: planetary precession and a secular change in obliquity. The effect of planetary precession is entirely in right ascension.

In summary, the secular effects on the orientations of the equatorial plane are:

- luni-solar precession,
- planetary precession, and
- a secular change in obliquity.

As is the convention, all of these secular effects are considered under the heading "precession." The periodic effects are:

- nutation in longitude, and

- nutation in obliquity.

3.6.1 Precession

The precession of coordinates from the mean equator and equinox of one epoch t_0 to the mean equator and equinox of t_1 is accomplished very simply. Examine Figure 3.6-1 and consider a position described by the vector $\bar{X}$ in the X_1, X_2, X_3 coordinate system that is defined by the mean equator and equinox of t_0. Likewise, consider the same position as described by the vector $\bar{Y}$ in the Y_1, Y_2, Y_3 system defined by the mean equator and equinox of t_1. The expression relating these vectors,

$$\bar{Y} = R_3(-z)R_2(\theta)R_3(-\zeta)\bar{X} \qquad (3.6 - 1)$$

follows directly from inspection of Figure 3.6-1.

It should be observed that 90° - ζ is the right ascension of the ascending node of the equator of epoch t_0 reckoned from the equinox of t_0, that 90° + Z is the right ascension of the node reckoned from the equinox of t_1, and that θ is the inclination of the equator of t_1 to the equator t_0. Numerical expressions for these rotation angles z, θ, ζ, were derived by Simon Newcomb, based partly upon theoretical considerations but primarily upon actual observation. (See References for the derivations.) The formulae used in GEODYN are relative to an initial epoch of 1950.0:

$$z = .^R30595320465 \times 10^-6d + .^R1097492 \times 10^-14d^2 + .^R178097 \times 10^-20d^3 \qquad (3.6 - 2)$$

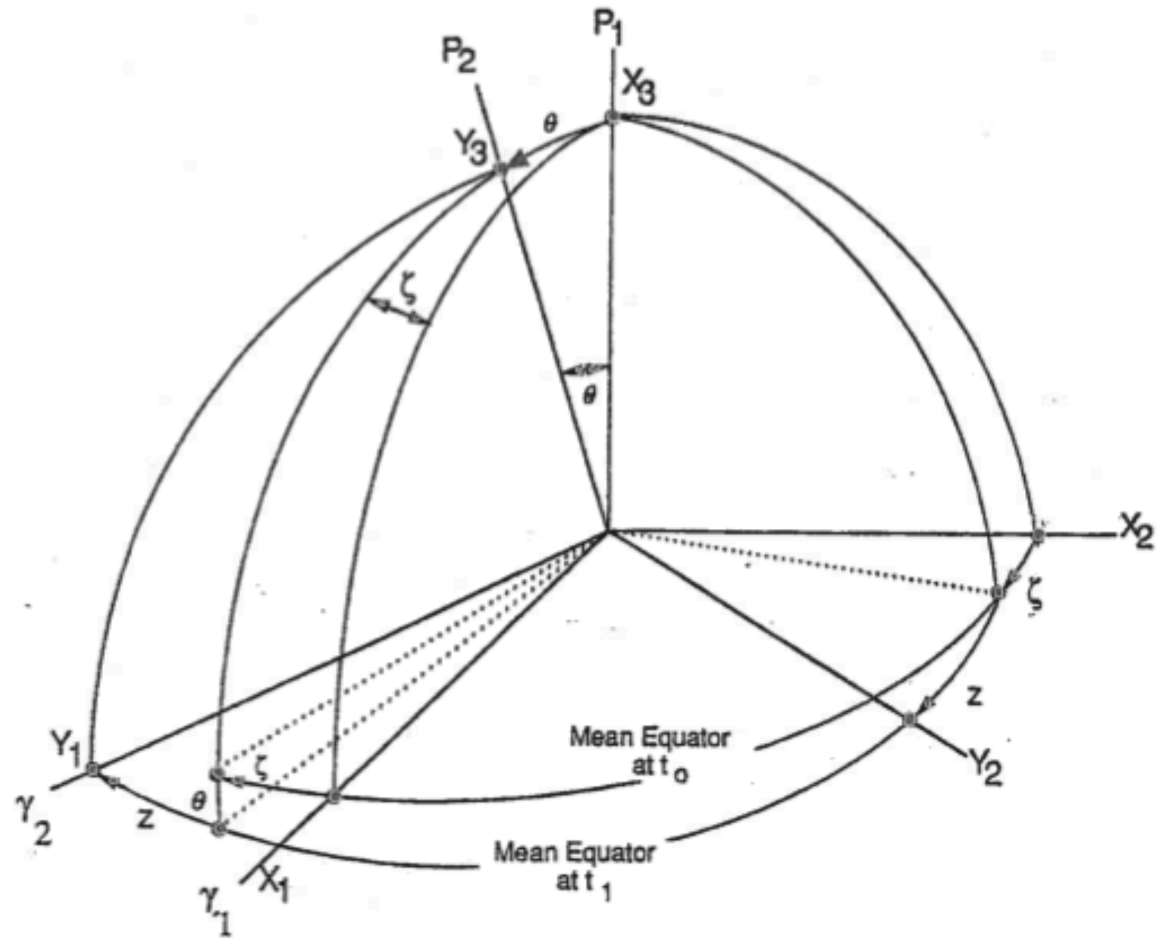
$$z = .^R30595320465 \times 10^-6d + .^R3972049 \times 10^-14d^2 + .^R191031 \times 10^-20d^3 \qquad (3.6 - 3)$$

$$q = .^R26603999754 \times 10^-6d - .^R1548118 \times 10^-14d^2 - .^R413902 \times 10^-20d^3 \qquad (3.6 - 4)$$

The angles are in radians. The quantity d is the number of elapsed days since 1950.0.

Figure 3.6-1 Precession: Rotation Between Mean Equator and Equinox of Epoch t_0 and Mean Equator and Equinox at Epoch t_1

P_1 = Direction of Mean Axis of Motion at t_0
P_2 = Direction of Mean Axis of Motion at t_1
γ_1 = Direction of Mean Equinox at t_0
γ_2 = Direction of Mean Equinox at t_1



P1
P2
X3
Y3
θ
ζ
θ
X2
ζ
z
Mean Equator
at t o
Y2
Y1
γ2
z
θ
ζ
Mean Equator
at t 1
X1
γ1

3.6.2 Nutation

The nutation of coordinates between mean and true equator and equinox of date is readily accomplished using rotation matrices. Examine Figure 3.6-2 and consider a position described by the vector $\bar{X}$ in the X_1, X_2, X_3 system which is described by the mean equator and equinox of date. Likewise, consider the same position as described by the vector $\bar{Z}$ in the Z_1, Z_2, Z_3 system defined by the true equator and equinox of date. The expression relating these vectors,

$$\bar{Z} = R_1(-\varepsilon_\tau)R_3(-\Delta\psi)R_1(\varepsilon_m)\bar{X} \qquad (3.6 - 5)$$

follows directly from inspection of Figure 3.6-2.

The definitions of these angles are:

- ε_τ - true obliquity of date
- ε_m - mean obliquity of date
- $\Delta\psi$ - nutation in longitude.

Note that $\Delta\varepsilon = \varepsilon_\tau$ - ε_m is the nutation in obliquity.

The remaining problem is to compute the nutation in longitude and obliquity. The algorithm used in GEODYN II was developed by Woolard.

The angles $\Delta\psi$ and $\Delta\varepsilon$ are obtained from the JPL ephemeris tape by the method described in Section 4.1. When the DE-96 or DE-118 planetary ephemeris is used, ε_m is calculated according to the following equation developed by Woolard [3.1 - 3.4].

$$\varepsilon_m = {}^r.409319755205 - {}^r.000000006217959d - {}^r.000021441 \times 10^{-12}d^2 + {}^r.180087 \times 10^{-21}d^3 \qquad (3.6 - 6)$$

where d = no. of ephemeris days elapsed from 1900 January .5 ephemeris time to the desired date.

When the DE-200 planetary ephemeris is used, the following equation is used for the calculation of ε_m

$$\varepsilon_m = 84381.''448 - 46''.8150T - 0.''001813T^3 \qquad (3.6 - 7)$$

where

T = Julian Ephemeris Centuries since J2000.0

J2000.0 = JED 2451545.0

Figure 3.6-2 Nutation: Rotation Between Mean Equator and Equinox of Date and True Equator and Equinox of Date.
ε_m = Mean Obliquity of Date
ε_T = True Obliquity of Date
γ_m= Direction of Mean Equinox of Date
γ_T = Direction of Time Equinox of Date

3.7 THE TRANSFORMATION TO THE EARTH CENTERED FIXED ADOPTED POLE COORDINATE SYSTEM

To accurately calculate the forces on an Earth satellite, it is necessary to take into account the change in the position of the Earth's geopotential field in inertial space due to the wandering of the pole. To account for this effect, GEODYN evaluates the forces on the satellite in the Earth centered fixed true pole (ECFT) coordinate system. The transformation from the inertial coordinate system (I) of the satellite to the ECFT system is accomplished as follows.

We must first transform from inertial coordinates to Earth centered fixed (ECF) coordinates by rotating about the Z axis by θ_g, the Greenwich hour angle. This rotation is given by the rotation matrix $R(\theta_g\ )$where:

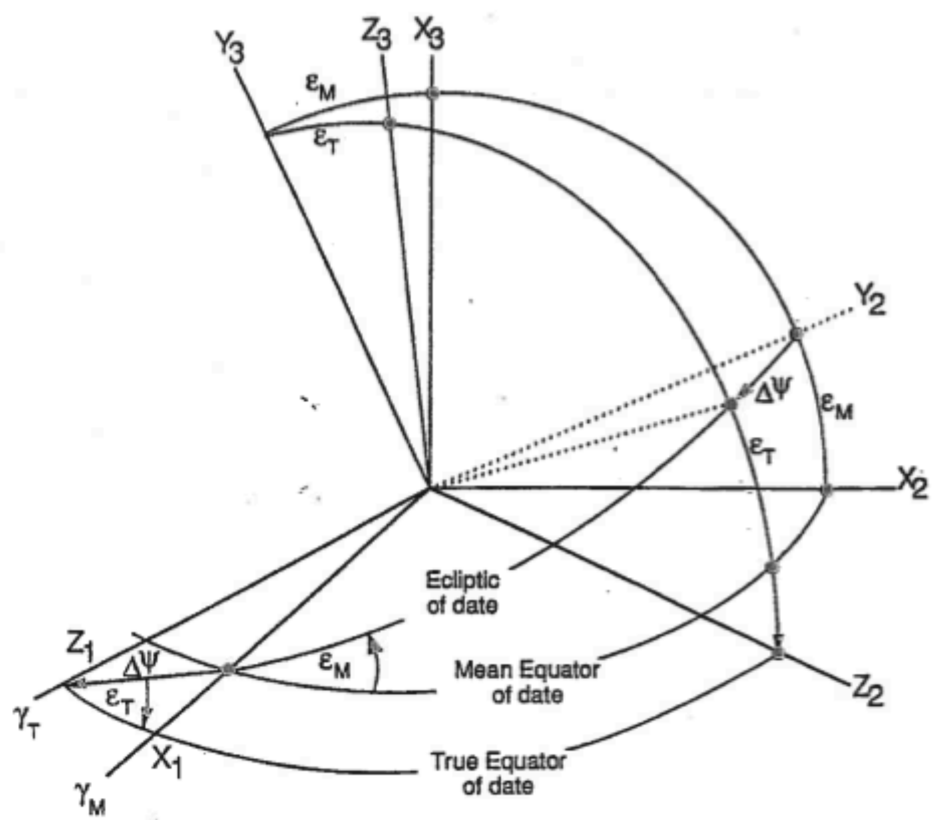
$$R(\theta_g) = \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 \\ -\sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.7 - 1)$$

If x and y are the coordinates of the instantaneous pole relative to the adopted pole, and are measured in seconds of arc (c.f. Figure 5.4-1, Section 5.4.1), then the transformation of a vector $\bar{X}_{ECFI}$ in the Earth centered fixed instantaneous pole to $\bar{X}_{ECFA}$ in the Earth centered fixed adopted pole is given by:

$$\bar{X}_{ECFA} = \begin{bmatrix} \cos x & 0 & \sin x \\ 0 & 1 & 1 \\ -\sin x & 0 & \cos x \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{bmatrix} \bar{X}_{ECFI} \qquad (3.7 - 2)$$

Which can be simplified to:

$$\bar{X}_{ECFA} = \begin{bmatrix} 1 & xy & x \\ 0 & 1 & -y \\ -x & y & 1 \end{bmatrix} \bar{X}_{ECFI} \qquad (3.7 - 3)$$


Y3
Z3
X3
εM
εT
Y2
ΔΨ
εM
εT
X2
Ecliptic
of date
Z1
ΔΨ
εM
εT
γT
Mean Equator
of date
Z2
X1
True Equator
of date
γM

Thus the transformation of a vector $\bar{X}1$ in inertial true of date coordinates to $\bar{X}_{ECFA}$ in Earth centered fixed adopted pole coordinates can be expressed as:

$$\bar{X}_{ECFA} = \begin{bmatrix} 1 & xy & x \\ 0 & 1 & -y \\ -x & y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 \\ -\sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{X}_1 \qquad (3.7 - 4)$$

4 LUNI-SOLAR-PLANETARY EPHEMERIS

4.1 EPHEMERIS USED IN GEODYN VERSIONS 7612.0 AND LATER

GEODYN uses the JPL export ephemeris for nutations, positions, and velocities of the Moon, Sun, and planets. The JPL export ephemeris package is an encoded tape that contains two files. The first program on file one, BCDEPH, reads the encoded ephemeris information on file two and writes the JPL binary ephemeris tape. The remaining information on file one is the software to read and interpolate the binary ephemeris tape, and is not used for GEODYN.

For GEODYN 7603 and earlier versions, a separate ephemeris tape generator program was used to read and interpolate the JPL ephemeris binary tape. The ephemeris tape generator program then precessed and nutated the coordinates from the JPL reference of mean equator and equinox of 1950.0 to true of date coordinates, and wrote the Everett interpolation coefficients for the coordinates onto a tape or disk. This was the ephemeris data read by GEODYN.

Two changes have been made in the above procedure in GEODYN 7610:

1. GEODYN reads the JPL binary ephemeris tape directly so no tape generator program is needed.
2. GEODYN interpolates for the ephemeris in the mean of 1950.0 reference system. This gives greater accuracy than interpolating in a true of date reference system because the high-frequency perturbations due to nutations are absent. After interpolation, the coordinates are then precessed and nutated to true of date.

A search is performed in the JPL binary ephemeris tape for the data records needed to cover the time span of the GEODYN run, and these are written onto a direct access disk. When ephemeris information is required, the program reads from the direct access disk the two consecutive ephemeris data records that span the required time interval.

4.1.1 Chebyshev Interpolation

Presently, the quantities being interpolated for are the nutation angles, $\Delta\varepsilon$ and $\Delta\psi$, and the position and velocity of Venus, Earth, Mars, Jupiter, Saturn, the Sun, and the Moon. For any one component (X, Y, Z of a particular body or $\Delta\varepsilon$ and $\Delta\psi$), there are NC number of Chebyshev coefficients per interval on the ephemeris data record. The ephemeris component desired, f, is calculated by the use of the following expansion:

$$f(t_c) = \sum_{K=0}^{NC-1} C_K P_K(t_c) \qquad (4.1 - 1)$$

where the C_K's are the Chebyshev coefficients for position read from the data record, and the $P_K(t_c)$'s are the Chebyshev polynomials for position defined as:

$$P_K(t_c) = \cos(K \cos{}^{-}1 t_c) \qquad (4.1 - 2)$$

The recurrence relationship

$$P_{K+1}(t_c) = 2t_c P_K(t_c) - P_{K-1}(t_c) \qquad (4.1 - 3)$$

holds and is used with equation (4.1-2) to calculate the polynomials. t_c is the Chebyshev time and is also the time t normalized between -1 and +1 on the time interval covered by the coefficients.

For any time t, three different values of t_c must be calculated since it is dependent on the time span covered by the coefficients for the particular body. The coefficients for Venus and the outer planets cover a 32-day time interval. For the Moon, the coefficients cover a four-day time interval, and for the Earth-Moon barycenter, the Sun, and nutations, the coefficients cover a 16-day time interval. Therefore, at every time t when the ephemerides and nutations are required, three sets of Chebyshev polynomials must be calculated.

The nutations are interpolated by using equation (4.1-1). The positions and velocities of the bodies are interpolated by using equation (4.1-1) as well. Velocity is calculated by differentiating equation (4.1-1) to get:

$$\frac{df(t_c)}{dt} = \sum_{K=1}^{NC-1} C_K V_K(t_c) \qquad (4.1 - 4)$$

where

$$V_K(t_c) = \frac{dP_K}{dt}(t_c)$$

The formulas for V_K are calculated by differentiating equation (4.1-3):

$$V_1(t_c) = 1$$
$$V_2(t_c) = 4t_c$$

$$V_{K+1}(t_c) = 2t_c V_K(t_c) + 2P_K(t_c) - V_{K-1}(t_c) \qquad (4.1 - 5)$$

Equation (4.1-4) could easily be differentiated again to compute accelerations. This is not presently done in GEODYN.

4.1.2 Coordinate Transformation

After interpolation, the positions and velocities of the planets, the Earth-Moon barycenter, and the Sun are known relative to the solar system barycenter and the Moon coordinates are known relative to the Earth. All vectors are referenced to the Earth mean equator and equinox of 1950.0. For further computations, GEODYN requires the coordinates to be true of date and geocentric.

The planet and sun vectors are transformed to geocentric in the following manner:

Given:

- $\bar{R}_{P_{SB}}$ the position of the planet relative to the solar system barycenter
- $\bar{R}_{EM_{SB}}$the position of the Earth-Moon barycenter relative to solar system barycenter
- G_M, the mass of the Moon
- G_E, the mass of the Earth
- $\bar{R}_{EM}$the geocentric position of the Moon

The geocentric position of the planet, $\bar{R}_{P_G}$ can be calculated as

$$\bar{R}_{P_G} = \left[\frac{G_M + G_K}{G_M}\right] \times \bar{R}_{EM} - \bar{R}_{EMSB} + \bar{R}_{P_{SB}} \qquad (4.1 - 6)$$

After using equation (4.1-6) to transform the planets to a geocentric coordinate system, precession and nutation are applied to transform from geocentric coordinates to true of date.

The matrix *RM2T*, used to rotate the vectors from mean of 1950.0 to true of date, and the equation of the equinoxes, *Eqn*, are calculated. Computing exact values for *RM2T(t)* and *Eqn(t)* requires calculating the trigonometric functions of the precession and nutation angles for every time t for which the ephemerides and/or the equation of the equinoxes are needed. This requires too much computer time, so the following method is used.

The exact values *RM2T* and *Eqn* are calculated at half-day intervals t_1 and t_2. For these times, the trigonometric functions of the angles are computed.

$$RM2T(t_1) = R_1(-\varepsilon_{t_1})R_3(-\Delta\psi_1)R_1(\varepsilon_{M_1})R_3(-Z_1)R_2(\theta_1)R_3(-\zeta_1) \quad (4.1 - 7)$$

$$RM2T(t_2) = R_1(-\varepsilon_{t_2})R_3(-\Delta\psi_2)R_1(\varepsilon_{M_2})R_3(-Z_2)R_2(\theta_2)R_3(-\zeta_2) \quad (4.1 - 8)$$

$$Eqn(t_1) = \Delta\psi_1 \cos\varepsilon_{t_1} \quad (4.1 - 9)$$

$$Eqn(t_2) = \Delta\psi_2 \cos\varepsilon_{t_2} \quad (4.1 - 10)$$

where R_1, R_2, and R_3 are the rotation matrices around the three principal axes of the coordinate system.

ε_t = true obliquity of date

ε_M = mean obliquity of date

$\Delta\psi$ = nutation in longitude

θi = inclination of the equator of t_1 to the equator of t_0

$90° - Z_1$ = right ascension of the ascending node of the equator of epoch t_0 reckoned from the equinox of t_1

$90° - Z_1$ = right ascension of the ascending node of the equator of epoch t_0 reckoned from the equinox of t_0

RM2T(t) can be calculated from knowing the rotation matrices for the angles at t_1 and the rotation matrices for the change in the angles from t_1 to t.

$$RM2T(t) = R_1(-\varepsilon_{t_1}-\alpha_1)R_3(-\Delta\psi_1-\beta_1)R_1(\varepsilon_{M_1}+\gamma_1)R_3(-Z_1-\delta_1)R_2(\theta_1+\lambda_1)R_3(-\zeta_1-\mu_1)$$

Since R_1, R_2, and R_3 are rotation matrices, the above equation can be rewritten as

$$RM2T(t) = R_1(-\varepsilon_{t_1})R_1(-\alpha_1)R_3(-\Delta\psi_1)R_3(-\beta_1)R_1(\varepsilon_{M_1})R_1(\gamma_1)R_3(-Z_1) \times R_3(-\delta_1)R_2(\theta_1)R_2(\lambda_1)R_3(-\zeta_1)R_3(-\mu_1) \quad (4.1 - 11)$$

where
$\alpha_1 = \varepsilon_t(t)$ - $\varepsilon_(t_1)$

$\beta_1 = \Delta\psi(t)$ - $\Delta\psi_1$
$\gamma_1 = \varepsilon_M(t)$ - ε_{M_1}
$\delta_1 = Z(t)$ - Z_1
$\lambda_1 = \theta(t)$ -θ_1
$\mu_1 = \zeta(t)$ - ζ_1

For t close enough to t_1, small angle approximations can be used to approximate the rotation matrices for α_1, β_1, γ_1, δ_1, λ_1, and μ_1

$$\sin\alpha_1 \simeq \alpha_1$$
$$\cos\alpha_1 \simeq 1$$

Therefore, using the small angle approximations and equation 4.1-11, a value for $RM2T(t)_1$ can be computed by calculating the actual values for the trigonometric functions of the nutation and precession angles at time t_1, and by calculating just the values of the angles at time t. To improve the accuracy of the approximation of $RM2T(t)$, $RM2T(t)_2$ is calculated by using the value of the rotation matrices at time t_2, and by using the change in the precession and nutation angles from t to t_2, where

$$RM2T(t)_2 = R_1(-\varepsilon_{t_2})R_1(-\alpha_2)R_3(-\Delta\psi_2)R_3(-\beta_2)R_1(\varepsilon_{M_2})R_1(\gamma_2)R_3(-Z_2)R_3(-\delta_2) \times R_2(\theta_2)R_2(\lambda_2)R_3(-\zeta_2)R_3(-\mu_2) \quad (4.1 - 12)$$

The $RM2T(t)$used is calculated as the weighted average of $RM2T(t)_1$ and $RM2T(t)_2$:

$$RM2T(t) = S \times RM2T(t)_2 + (1 - S) \times RM2T(t)_1 \quad (4.1 - 13)$$

where

$$S = \frac{t - t_1}{t_2 - t_1}$$

A similar method is used to approximate $Eqn(t)$

$$Eqn(t)_1 = \Delta\psi(t)\cos(\varepsilon_{t_1} + \alpha_1)$$
$$\simeq \Delta\psi(t)[\cos\varepsilon_{t_1} - \alpha_1 \sin\varepsilon_{t_1}]$$

$$Eqn(t)_2 \simeq \Delta\psi(t)\cos(\varepsilon_{t_2} - \alpha_2 \sin\varepsilon_{t_2})$$
$$Eqn(t) \simeq S \times Eqn(t)_2 + (1 - S) \times Eqn(t)_1 \quad (4.1 - 14)$$

5 THE OBSERVER

This section is concerned with the position and coordinate systems of the observer. Thus it will cover

- geodetic station position coordinates
- topocentric coordinate systems,
- time reference systems, and
- polar motion.

The geodetic station position coordinates are a convenient and quite common way of describing station positions. Consequently, GEODYN contains provisions for converting to and from these coordinates, including the transformation of the covariance matrix for the determined Cartesian station positions.

The topocentric coordinate systems are coordinate systems to which the observer references his observations.

The time reference systems are the time systems in which the observer specifies his observations. The transformations between time reference systems are also given. These latter are used both to convert the observation times to ET time, which is the independent variable in the equations of motion, and to convert the GEODYN output to UTe time, which is the generally recognized system for output.

The positions of the observers in GEODYN are referred to an Earth.fixed system defined by the mean pole implied by the supplied (or adjusted) polar motion series and the frame realized by the station coordinates themselves. They are rotated into the Earth-fixed system of date at each observation time by applying "polar motion," which is considered to be slippage of the Earth's crust.

5.1 GEODETIC COORDINATES

Frequently, it is more convenient to define the station positions in a spherical coordinate system. The spherical coordinate system uses an oblate spheroid or an ellipsoid of revolution as a model for the geometric shape of the Earth. The Earth is flattened slightly at the poles and bulges a little at the equator; thus, a cross-section of the Earth is approximately an ellipse. Rotating an ellipse about its shorter axis forms an oblate spheroid.

An oblate spheroid is uniquely f, where $f = \frac{a-b}{a}$ (see Figure 5.1-1).

This model is used in the GEODYN system. The spherical coordinates utilized are termed geodetic coordinates and are defined as follows:

- ϕ is geodetic latitude, the acute angle between the semi-major axis and a line through the observer perpendicular to the spheroid.
- λ is east longitude, the angle measured eastward in the equatorial plane between the Greenwich meridian and the observer's meridian.
- h is spheroid height, the perpendicular height of the observer above the reference spheroid.

Consider the problem of converting from ϕ, λ and h to X_e, Y_e, and Z_e, the Earth-fixed Cartesian coordinates. The geometry for an X-Z plane is illustrated in Figure 5.1-1. The equation for this ellipse is

$$X^2 + \frac{Z^2}{1 - e^2} = a^2 \tag{5.1 - 1}$$

where the eccentricity has been determined from the flattening of the familiar relationship

$$e^2 = 1 - (1 - f)^2 \tag{5.1 - 2}$$

The equation for the normal to the surface of the ellipse yields

$$\tan\phi = \frac{-dX}{dZ} \tag{5.1 - 3}$$

By taking differentials on equation (5.1-1) and applying the result in equation (5.1-3), we arrive at

$$\frac{Z}{X} = (1 - e^2)\tan\phi \tag{5.1 - 4}$$

The simultaneous solution of equations (5.1-1) and (5.1-4) for X yields:

$$X = \frac{a\cos\phi}{\sqrt{1 - e^2\sin^2\phi}} \tag{5.1 - 5}$$

From inspection of Figure 5.1-1 we have:

$$\cos\phi = \frac{X}{N} \tag{5.1 - 6}$$

and hence, applying equation 5.1-5,

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\phi}} \tag{5.1 - 7}$$

For an observer at a distance h from the reference ellipsoid, the observer's coordinates (X I Z) become

$$X = N\cos\phi + h\cos\phi \tag{5.1 - 8}$$

and

$$Z = N(1-e^2)\sin\phi + h\sin\phi \tag{5.1 - 9}$$

The conversion of ϕ, λ, and h to X_e, Y_e, and Z_e is then

$$\begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix} = \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ (N+h-e^2N)\sin\phi \end{bmatrix} \tag{5.1 - 10}$$

FIgure 5.1-1 Diagram of Geodetic and Geocentric Latitudes

'The problem of converting from X_e, Y_e, and Z_e to ϕ, λ and h is more complex as we cannot start with a point on the reference ellipsoid. For this reason the determination of accurate values for ϕ and h requires an iterative technique.

Conversion to Geodetic Coordinates

For the problem of converting station coordinates in X_e, Y_e, and Z_e toϕ ,λ, and h we know that N is on the order of magnitude of an Earth radius, and h is a few meters. Hence

$$h << N \tag{5.1 - 11}$$

The Earth is approximately a sphere, hence

$$e << 1 \tag{5.1 - 12}$$

Therefore, again working in our X-Z plane (see Figure 5.1-1),

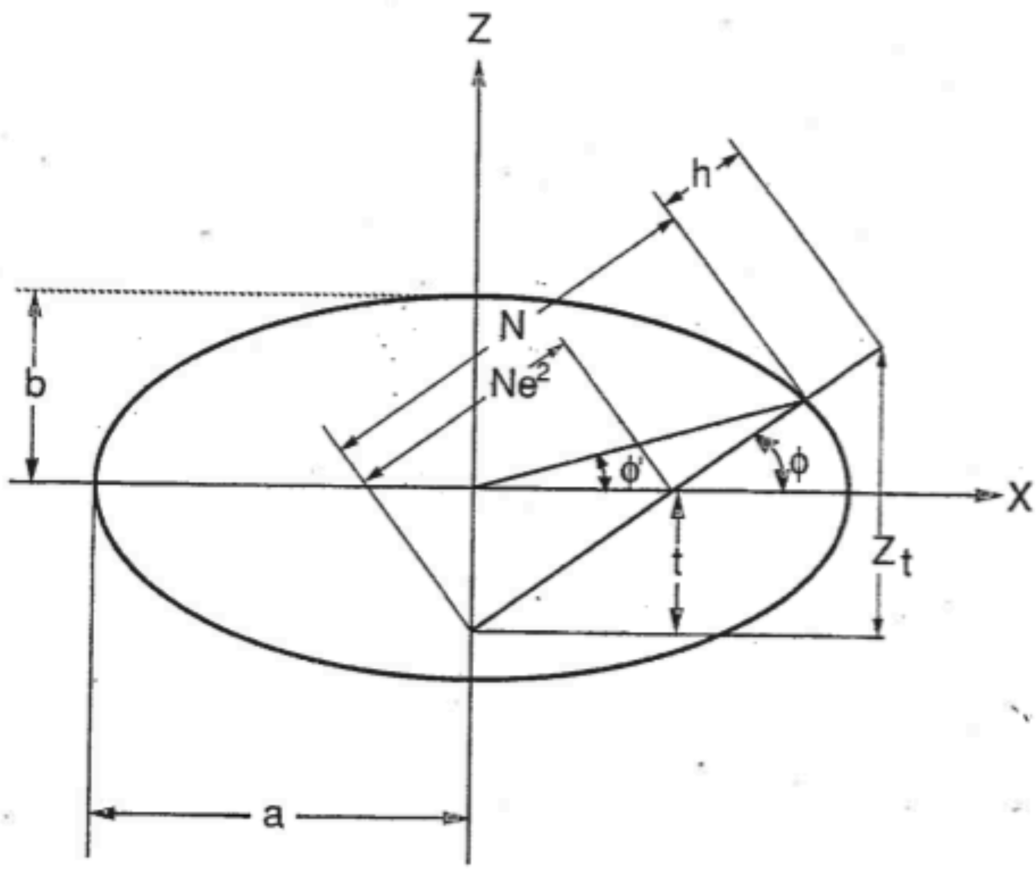
$$N\sin\phi \simeq Z \tag{5.1 - 13}$$

From Figure 5.1-1 (see also equation (5.1-9)) we have

$$t = Ne^2\sin\phi \tag{5.1 - 14}$$

or, for an initial approximation,

$$t \simeq e^2 Z \tag{5.1 - 15}$$



Z
h
N
Ne2
b
ϕ'
ϕ
X
t
Zt
a

The series of calculations to be performed on each iteration is:

$$Z_t = Z + t \tag{5.1 - 16}$$

$$N + h = [X_e^2 + Y_e^2 + Z_e^2]^{\frac{1}{2}} \tag{5.1 - 17}$$

$$\sin\phi = \frac{Z_t}{(N + h)} \tag{5.1 - 18}$$

$$N = \frac{a}{[1 - e^2 \sin^2\phi]}^{\frac{1}{2}} \tag{5.1 - 19}$$

$$t = Ne^2 \sin\phi \tag{5.1 - 20}$$

When t converges, ϕ and h are computed from $\sin\phi$ and $(N + h)$. The computation of λ is obvious; it being simply

$$\lambda = \tan^{-1}\left[\frac{Y_e}{X_e}\right] \tag{5.1 - 21}$$

A less rigorous procedure is used to evaluate the satellite height for atmospheric density computations. Because $e << 1$, we may write an approximation to equation (5.1-9):

$$Z = (N + h)(1 - e^2)\sin\phi = Z_e \tag{5.1 - 22}$$

From Figure 5.11,

$$X = (N + h)\cos\phi = \sqrt{X_e^2 + Y_e^2} \tag{5.1 - 23}$$

and by remembering equation (5.1-2),

$$\phi = \tan^{-1}\left[\frac{Z_e}{(1-f)^2\sqrt{X_e^2+Y_e^2}}\right] \tag{5.1 - 24}$$

The equation for the ellipse, equation (5.1-1), yields the following formula for the radius of the ellipsoid:

$$r_{ellipsoid} = \sqrt{X^2 + Y^2} = \frac{a(1 - f)}{\sqrt{1 - (2f - f^2)(1 - \sin\phi')}} \tag{5.1 - 25}$$

where ϕ' is the geocentric latitude. After applying the Binomial Theorem, we arrive at

$$r_{ellipsoid} = a\left[1 - (f + \tfrac{3}{2}f^2)\sin^2\phi' + \tfrac{3}{2}f^2\sin^4\phi'\right] \tag{5.1 - 26}$$

wherein terms on the order of P have been neglected. The (spheroid) height may then be calculated from r , the geocentric radius of the satellite:

$$h = r - r_{(ellipsoid)} \tag{5.1 - 27}$$

or

$$h = \sqrt{X_e^2 + Y_e^2 + Z_e^2} - a + (af + \frac{3}{2}af^2)\sin^2\phi' - \frac{3}{2}af^2\sin^4\phi' \qquad (5.1 - 28)$$

The sine of the geocentric latitude, $\sin\phi'$ is of course $\frac{Z_e}{r}$. Also required are the partial derivatives of h with respect to position for the drag variational partials computations:

$$\frac{\partial h}{\partial r_i} = \frac{r_i}{r} + 2\sin\phi' \left[\left[af + \tfrac{3}{2}af^2 \right] - 3af^2\sin^2\phi' \right] \left[\tfrac{Z_e r_i}{r^3} + \tfrac{1}{r}\tfrac{\partial Z_e}{\partial r_i} \right] \qquad (5.1 - 29)$$

where the r_i are the Earth-fixed components of $\bar{r}$; i.e., X_e, Y_e, Z_e. . In addition to the conversion of the coordinates themselves, GEODYN also converts covariance matrices for the station positions to either the ϕ, λ, h system or the Earth-fixed rectangular system.

$$V_{OUT} = P^T V_{IN} P \qquad (5.1 - 30)$$

where V_{OUT} is the output covariance matrix, V_{IN} is the input covariance matrix, and P is the matrix of partials relating the coordinates in the output system to the coordinates in the input system. These partial derivatives (in P) which GEODYN requires are for X_e, Y_e, Z_e with respect to ϕ, λ h and vice versa. These partials are:

$$\begin{aligned}
\frac{\partial\phi}{\partial X_e} &= \frac{-X_e Z_e(1-e^2)}{((1-e^2)^2(X_e^2+Y_e^2)+Z_e^2)(X_e^2+Y_e^2)^{\frac{1}{2}}} \\
\frac{\partial\phi}{\partial Y_e} &= \frac{-Y_e Z_e(1-e^2)}{((1-e^2)^2(X_e^2+Y_e^2)+Z_e^2)(X_e^2+Y_e^2)^{\frac{1}{2}}} \\
\frac{\partial\phi}{\partial Z_e} &= \frac{(X_e^2+Y_e^2)(1-e^2)}{((1-e^2)^2(X_e^2+Y_e^2)+Z_e^2)(X_e^2+Y_e^2)^{\frac{1}{2}}} \\
\frac{\partial\lambda}{\partial X_e} &= -\frac{Y_e}{(X_e^2+Y_e^2)} \\
\frac{\partial\lambda}{\partial Y_e} &= -\frac{X_e}{(X_e^2+Y_e^2)} \qquad (5.1 - 31) \\
\frac{\partial\lambda}{\partial Z_e} &= 0
\end{aligned}$$

$$\frac{\partial h}{\partial X_e} = \frac{\partial \phi}{\partial X_e}(\frac{-e^2 a(1-e^2)(\sin\phi\cos\phi)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} - \frac{Z_e\cos\phi}{\sin^2\phi})$$

$$\frac{\partial h}{\partial Y_e} = \frac{\partial \phi}{\partial Y_e}(\frac{-e^2 a(1-e^2)(\sin\phi\cos\phi)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} - \frac{Z_e\cos\phi}{\sin^2\phi})$$

$$\frac{\partial h}{\partial Z_e} = \frac{\partial \phi}{\partial Z_e}(\frac{-e^2 a(1-e^2)(\sin\phi\cos\phi)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} - \frac{Z_e\cos\phi}{\sin^2\phi}) + \frac{1}{\sin\phi}$$

$$\frac{\partial X_e}{\partial \phi} = -\sin\phi\cos\lambda\left[N + h - \frac{Ne^2\cos^2\phi}{1-e^2\sin^2\phi}\right]$$

$$\frac{\partial X_e}{\partial \lambda} = -(N+h)\cos\phi\sin\lambda$$

$$\frac{\partial X_e}{\partial h} = \cos\phi\cos\lambda$$

$$\frac{\partial Y_e}{\partial \phi} = -\sin\phi\sin\lambda\left[N + h - \frac{Ne^2\cos^2\phi}{1-e^2\sin^2\phi}\right]$$

$$\frac{\partial Y_e}{\partial \lambda} = (N+h)\cos\phi\cos\lambda \qquad (5.1 - 32)$$

$$\frac{\partial Y_e}{\partial h} = \cos\phi\sin\lambda$$

$$\frac{\partial Z_e}{\partial \phi} = -\cos\phi\left[h + N(1-e^2)\left[1 + \frac{e^2\sin^2\phi}{1-e^2\sin^2\phi}\right]\right]$$

$$\frac{\partial Z_e}{\partial \lambda} = 0$$

$$\frac{\partial Z_e}{\partial h} = \sin\phi$$

5.1.1 Station Constraints

In GEODYN, it is possible to designate a tracking station as one of two types: a master station or a constrained station. When the program iterates to compute a better orbit, it can adjust the position of a master station in order to better fit the data. The position of the constrained station will be changed as a result of the adjustment in the position of the master station to which it is constrained. The amount and direction of the change in the constrained station's position is a function of the change in the master's position. This function is determined by the nature of the constraints linking the constrained station to the master station.

A different set of constraints. is used for each system of coordinates: cartesian, cylindrical and geodetic spherical.

In cartesian coordinates, the form of the constraints is: X_m - X_c = constant Y_m- Y_c = constant Z_m - Z_c = constant where the subscripts m and c refer to master and constrained stations, respectively. These constraints can be expressed as:

$$\bar{X}_{c_{NEW}} \approx \bar{X}_{c_{OLD}} + \bar{\delta}$$

where

$$\bar{\delta} = \bar{X}_{m_{NEW}} - \bar{X}_{m_{OLD}}$$

The constraint equations in the cylindrical system have the same form as the cartesian constraints:
ρ_m - ρ_c = constant
θ_m - θ_c = constant
Z_m - Z_c = constant

or, if $\bar{X} = (\rho, \theta, Z)$

$$\bar{X}_{c_{NEW}} = \bar{X}_{c_{OLD}} + \bar{\delta}$$

where

$$\bar{\delta} = \bar{X}_{m_{NEW}} - \bar{X}_{m_{OLD}}$$

The method of constraining stations in the spherical geodetic system is more complicated than those used for the cartesian and cylindrical cases. All the adjustments to the stations are done using geocentric spherical coordinates. The procedure used is to convert the unadjusted positions from geodetic to geocentric coordinates, make the position adjustments to the geocentric coordinates, and then convert the adjusted positions from geocentric to geodetic coordinates. The original position of the master is (ϕ_m, λ_m, R_m) and the original position of the constrained station is (ϕ_c, λ_c, R_c) both positions expressed in geocentric spherical coordinates. The adjustment of the master station latitude and longitude is done by first rotating the station around the spin axis of the earth by the amount $\Delta\lambda$, as shown in Figure 5.1-2. The new master position is (ϕ_m, $\lambda_m + \Delta\lambda$, R_m) and the new constrained position is (ϕ_c, $\lambda_c + \Delta\lambda$, R_c). To adjust the master in latitude by $\Delta\phi$ an axis is defined which lies in the equatorial plane and which is perpendicular to the plane determined by the new master station position (ϕ_m,λ_m+$\Delta\lambda$ R_m) and the north pole (see Figure 5.1-2). The longitude of the positive end of this axis is $\lambda_m + \Delta\lambda + 90°$. A rotation about this axis by $-\Delta\phi$ completes the adjustment of the master to its new coordinates (ϕ_m+$\Delta\phi$, λ_m

+ $\Delta\lambda$,R_m).
The new coordinates of the constrained station are:

$$\phi_c^* = \arcsin\{\sin(+\Delta\phi)[\cos(\lambda_m - \lambda_c)\cos(\phi_c)] + \cos(+\Delta\phi)\sin(\phi_c)\}$$

$$\begin{aligned}\tan\lambda_c^* = \{&(\tan(\lambda_c + \Delta\lambda) + \cos(+\Delta\phi) - 1)(\sin^2(\lambda_m + \Delta\lambda)\tan(\lambda_c + \Delta\lambda) + \\ &\sin(\lambda_m + \Delta\lambda)\cos(\lambda_m + \Delta\lambda)) + \sin(+\Delta\phi)(\sin(\lambda_m + \Delta\lambda)\tan(\phi_c)/\cos(\lambda_c + \Delta\lambda))\}/ \\ \{&1 + (\cos(+\Delta\phi) - 1)(\cos^2(\lambda_m + \Delta\lambda) + \sin(\lambda_m + \Delta\lambda))\cos(\lambda_m + \Delta\lambda)\tan(\lambda_c + \Delta\lambda)) + \\ &\sin(+\Delta\phi)(\cos(\lambda_m + \Delta\lambda)\tan(\phi_c)/\cos(\lambda_c + \Delta\lambda))\}\end{aligned}$$

The adjustment in radius is simply:

$$R_c^* = R_c + \Delta R$$

where

$$\Delta R = R_m^* - R_m$$

Figure 5.1-2 Adjustment of Constrained Stations

The final adjusted positions are:

MASTER (ϕ_m + $\Delta\phi$, λ_m + $\Delta\lambda$, R_m + ΔR)
CONSTRAINED (ϕ_c^*, λ_c^*, R_c + ΔR)

The partial derivatives of the measurements made at each station will also change when the station positions are adjusted. The partials for the constrained station can be computed from those of the master station using:

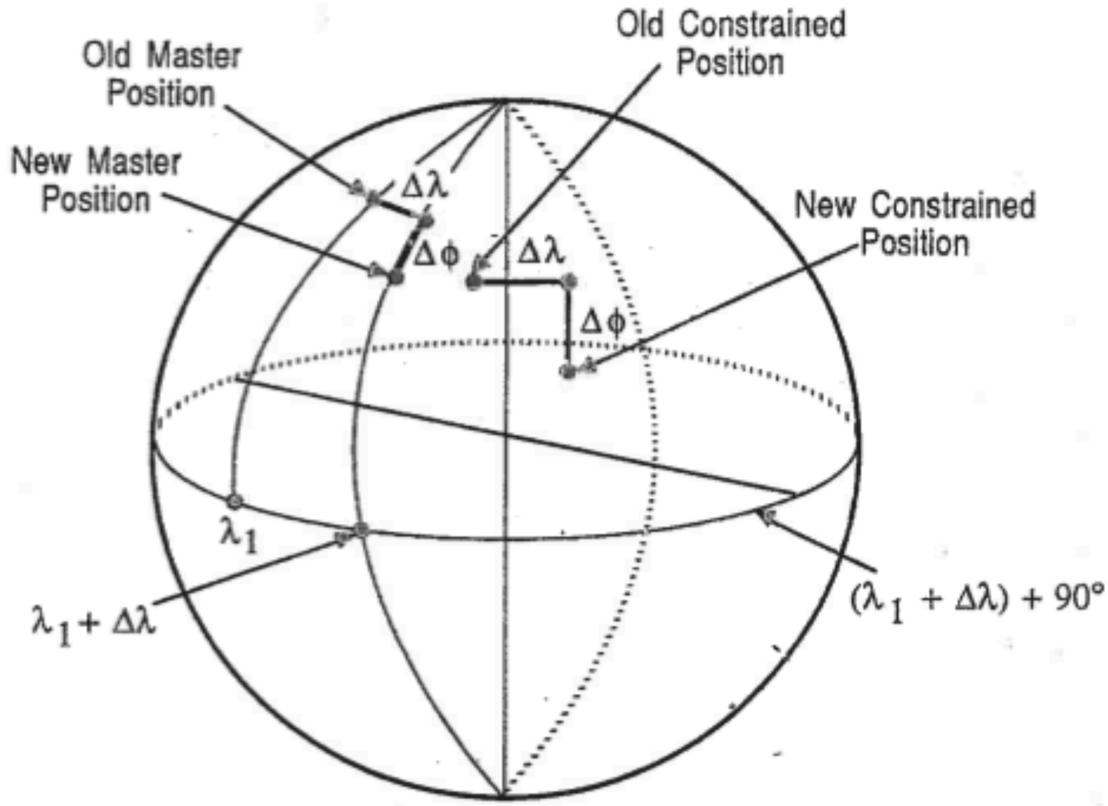
$$\frac{\partial M}{\partial X_{cj}} = \sum_i \frac{\partial M}{\partial X_{ml}} \frac{\partial X_{ml}}{\partial X_{cj}}$$

where M is some measurement (e.g., range, range rate)

The partials $\partial X_{cj}/\partial X_{mi}$ $\left[= 1 \ / \ \left[\frac{\partial X_{mi}}{\partial X_{cj}} \right] \right]$

can be computed from the constraints and are, in general, different for each system of constraint equations.

For both the cartesian and cylindrical constraint equations, the partials ∂X_{cj} / ∂X_{mi} can be compared directly from the constraint equations.


Old Master
Position
New Master
Position
Old Constrained
Position
New Constrained
Position
Δλ
Δφ
Δλ
Δφ
λ1
λ1 + Δλ
(λ1 + Δλ) + 90°

The matrix of cartesian partials is:

$$\begin{bmatrix} \frac{\partial X_c}{\partial X_m} & \frac{\partial Y_c}{\partial X_m} & \frac{\partial Z_c}{\partial X_m} \\ \frac{\partial X_c}{\partial Y_m} & \frac{\partial Y_c}{\partial Y_m} & \frac{\partial Z_c}{\partial Y_m} \\ \frac{\partial X_c}{\partial Z_m} & \frac{\partial Y_c}{\partial Z_m} & \frac{\partial Z_c}{\partial Z_m} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrix of cylindrical partials is:

$$\begin{bmatrix} \frac{\partial \rho_c}{\partial \rho_m} & \frac{\partial \theta_c}{\partial \rho_m} & \frac{\partial Z_c}{\partial \rho_m} \\ \frac{\partial \rho_c}{\partial \theta_m} & \frac{\partial \theta_c}{\partial \theta_m} & \frac{\partial Z_c}{\partial \theta_m} \\ \frac{\partial \rho_c}{\partial Z_m} & \frac{\partial Y_c}{\partial Z_m} & \frac{\partial Z_c}{\partial Z_m} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, the spherical case is more difficult. Using geocentric spherical coordinates, most of the partials can be computed easily. By inspection of the method described above for the spherical adjustment,

$$\frac{\partial R_c}{\partial R_m = 1}$$

and

$$\frac{\partial R_c}{\partial \phi_m} = \frac{\partial R_c}{\partial \lambda_m} = \frac{\partial \phi_c}{\partial R_m} = \frac{\partial \lambda_c}{\partial R_m} = 0$$

Also, because of the special way the station way adjustment is made, by first rotating by $\Delta\lambda$ we have

$$\frac{\partial \lambda_c}{\partial \lambda_m} = 1$$

and

$$\frac{\partial \phi_c}{\partial \lambda_m} = 0$$

The remaining partials, $\frac{\partial \lambda_c}{\partial \phi_m}$ and $\frac{\partial \phi_c}{\partial \phi_m}$, have been calculated by expanding the expressions for ϕ_c^* an λ_c^*, assuming that $\Delta\phi$ is small, and approximating $\sin(\Delta\phi) \simeq \Delta\psi$ and $\cos(\Delta\phi) \simeq 1$. This is a reasonable procedure since, in practice, the station adjustment is always on the order of a few seconds of arc. The result of this approximation has the form

$$\lambda_c^* = \lambda_c + f \equiv \lambda_c + \frac{\partial \lambda_c}{\partial \phi_m}\Delta\phi$$

$$\phi_c^* = \phi_c + g \equiv \phi_c + \frac{\partial \phi_c}{\partial \phi_m}\Delta\phi$$

f and g are obtained by subtracting λ_c and ϕ_c from the approximated forms of λ_c^* and ϕ_c^*. Thus

$$\frac{\partial \lambda_c}{\partial \phi} = \frac{f}{\Delta \phi} = \left[\frac{\sin(\lambda_m+\Delta\lambda)-\cos(\lambda_m+\Delta\lambda)\tan(\lambda_c+\Delta\lambda)}{1+\tan^2(\lambda_c+\Delta\lambda)}\right]\left[\frac{\tan\phi_c}{\cos(\lambda_c+\Delta\lambda)}\right]$$

$$\frac{\partial \phi_c}{\partial \phi_m} = \frac{g}{\Delta \phi} = \cos(\lambda_m - \lambda_c)$$

The matrix of spherical partial is:

$$\begin{bmatrix} \frac{\partial \phi_c}{\partial \phi_m} & \frac{\partial \lambda_c}{\partial \phi_m} & \frac{\partial R_c}{\partial \phi_m} \\ \frac{\partial \phi_c}{\partial \lambda_m} & \frac{\partial \lambda_c}{\partial \lambda_m} & \frac{\partial R_c}{\partial \lambda_m} \\ \frac{\partial \phi_c}{\partial R_m} & \frac{\partial \lambda_c}{\partial R_m} & \frac{\partial R_c}{\partial R_m} \end{bmatrix} = \begin{bmatrix} g/\Delta\phi & f/\Delta\phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The partials are computed and expressed in geocentric spherical, not geodetic spherical coordinates. This difference in coordinate systems should not change the partials too much, so that these partials may be chained to partials computed with respect to geodetic coordinates without substantial error.

5.1.2 Baseline Sigmas

GEODYN calculates the following set of partials:

$$P = \left[\frac{\partial baseline}{\partial X_{STA_i}}, \frac{\partial baseline}{\partial Y_{STA_i}}, \frac{\partial baseline}{\partial Z_{STA_i}}, \frac{\partial baseline}{\partial X_{STA_j}}, \frac{\partial baseline}{\partial Y_{STA_j}}, \frac{\partial baseline}{\partial Z_{STA_j}}\right]$$

To calculate the baseline sigma GEODYN uses:

$$\sigma^2_{baseline} = P^T V P$$

where V is the following variance / covariance matrix:

$$V = \begin{bmatrix} \sigma^2_{x_i} & Cx_iy_i & Cx_iz_i & Cx_ix_j & Cx_jy_j & Cx_iz_j \\ Cy_ix_i & \sigma^2_{y_i} & Cy_iz_i & Cy_ix_j & Cy_jy_j & Cy_iz_j \\ Cz_ix_i & Cz_iy_i & \sigma^2_{z_i} & Cz_ix_j & Cz_jy_j & Cz_iz_j \\ Cx_jx_i & Cx_jy_i & Cx_jz_i & \sigma^2_{x_j} & Cx_jy_j & Cx_jz_j \\ Cy_jx_i & Cy_jy_i & Cy_jz_i & Cy_ix_j & \sigma^2_{y_j} & Cy_jz_j \\ Cz_jx_i & Cz_jy_i & Cz_jz_i & Cz_jy_j & Cz_jy_j & \sigma^2_{z_j} \end{bmatrix}$$

The subscripts i and j refer to i^{th} and j^{th} stations respectively.

5.1.3 Station Velocities

The old Range model

$$R = [(x_s - x_T)^2 + (y_s - y_T)^2 + (z_s - z_T)^2]^{\frac{1}{2}}$$

where x_s, y_s, z_s are the satellite coordinates In a 3-D Cartesian Coordinate system and x_T, y_T, z_T are the tracking station coordinates in the same coordinate system

has been changed in GEODYN to accommodate for station linear velocities:

$$R = \{[(x_s - (x_{oT} + v_x\Delta t)]^2 + [(y_s - (y_{oT} + v_y\Delta t)]^2 + [(z_s - (z_{oT} + v_z\Delta t)]^2]\}^{\frac{1}{2}}$$

where:
msl.
x_{oT}, y_{oT}, z_{oT} are the tracking station coordinates at a reference epoch,
v_x, v_y, v_z are the velocity components in a 3-D Kartesian coordinate system and
Δt is the time difference between the velocity reference epoch and the run epoch start time.

The partial derivatives for the velocities are

$$\frac{\partial R}{\partial v_x} = \frac{\partial R}{\partial x_T}\frac{\partial x_T}{\partial v_x} = -\frac{x_s - x_T}{R}\Delta t$$
$$\frac{\partial R}{\partial v_x} = \frac{\partial R}{\partial x_T}\frac{\partial x_T}{\partial v_x} = -\frac{x_s - x_T}{R}\Delta t$$
$$\frac{\partial R}{\partial v_x} = \frac{\partial R}{\partial x_T}\frac{\partial x_T}{\partial v_x} = -\frac{x_s - x_T}{R}\Delta t$$

The partials are computed in subroutine RANGE and summed up in subroutine PSUM.

5.2 TOPOCENTRIC COORDINATE SYSTEMS

The observations of a spacecraft are usually referenced to the observer, and therefore an additional set of reference systems is used for this purpose. The origin of these systems, referred to as topocentric coordinate systems, is the observer on the surface of the earth. Topocentric right ascension and declination are measured in an inertial system whose Z axis and fundamental plane are parallel to those of the geocentric inertial system. The X axis in this case also points toward the vernal equinox.

The other major topocentric system is the Earth-fixed system determined by the zenith and the observer's horizon plane. This is an orthonormal system defined by $\hat{N}, \hat{E}, \hat{Z}$, which are unit vectors which point in the same directions as vectors from the observer pointing north, east, and toward the zenith. Their definitions are:

$$\hat{N} = \begin{bmatrix} -\sin\psi\cos\lambda \\ -\sin\psi\sin\lambda \\ \cos\psi \end{bmatrix} \tag{5.2 - 1}$$

$$\hat{E} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \tag{5.2 - 2}$$

$$\hat{Z} = \begin{bmatrix} \cos\psi\cos\lambda \\ \cos\psi\sin\lambda \\ \sin\psi \end{bmatrix} \tag{5.2 - 3}$$

where ψ is the geodetic latitude and λ is the east longitude of the observer (see Section 5.1). This latter system is the one to which such measurements as azimuth and elevation, X and Y angles, and direction cosines are related. It should be noted that the reference systems for range and range rate must be Earth-fixed, but the choice of origin is arbitrary. In GEODYN, range and range rate are not considered to be topocentric, but rather geocentric.

5.3 TIME REFERENCE SYSTEMS

Three principal time systems are currently in use: ephemeris time, atomic time, and universal time. Ephemeris time is the independent variable in the equations of motion of the spacecraft, the sun, and the planets; this time is the uniform mathematical time. The corrections that must be applied to universal time to obtain ephemeris time are published in the American Ephemeris and Nautical Almanac or alternatively by BIB, the "Bureau Intemational de l'Heure." Atomic time is a time based on the oscillations of the hyperfine ground state transition of cesium 133. In practice, A1 time is based on the mean frequency of oscillation of several cesium standards as compared with the frequency of ephemeris time. The relationship between ET and A1 is the following:

$$ET - A1 = 32.1496183$$

The following equation is used to calculate (UT2 - UT1) for any year:

$$(UT2 - UT1) = +0^s.022\sin 2\pi t - 0^s.012\cos 2\pi t - 0^s.006\sin 4\pi t + 0^s.007\cos 4\pi t \tag{5.3 - 1}$$

$t =$ fraction of the tropical year elapsed from the beginning of the Besselian year for which the calculation is made. (1 tropical year = 365.2422 days).

This difference, (UT2 - UT1), is also known by the name "seasonal variation." The time difference (Al - UTI), is computed by linear interpolation from a table of values. The spacing-for the table is every 10 days, which matches the increment for the "final time of emission" data published by the U. S. Naval Observatory in the bulletin, "Time Signals." The differences for this table are determined by

$$(A1 = UT1) = (A1 - UTC) - (UT1 - UTC)$$

The values for (UT1 - UTC) are obtained from "Circular D," BIH. The difference (A1 - UTC) are determined according to the following procedure.
The computation of (A 1 - UTC) is simple, but not so straightforward. UTC contains discontinuities both in epoch and in frequency because an attempt is made to keep the difference between a UTC clock and a UT2 clock less than 0.1 sec. When adjustments are made, by international agreement they are made in steps of 0.1 sec. and only at the beginning of the month; i.e., at $0^h.0$ UT of the first day of the month. The general formula which Is used to compute (A1 - UTC) is

$$(A1 - UTC) = a_o + a_1(t - t_0)$$

Both ao and at are recovered from tables. The values in the table for ao arc the values of (AI - UTe) at the time of each particular step adjustment. The values in the table for a] are the values for the new rates of change between the two systems after each step adjustment.

Values for a_o and a_1 are published both by the U. S. Naval Observatory and BIH.

5.3.1 Partial Derivatives

We require the partial derivatives of the measurement with respect to the (A1 - UT1) difference. The transformation of station position from earth-fixed coordinates to true of date inertial coordinate is accomplished by a simple rotation about the earth's spin axis by the Greenwich hour angle θ_g

$$\bar{w} = R(\theta_g)\bar{u} \qquad (5.3 - 2)$$

where $\bar{u}$ = station vector in earth-centered fixed coordinates. $\bar{w}$ = station vector in true of date inertial coordinates.

$$R = \begin{bmatrix} \cos\theta_g & -\sin\theta_g & 0 \\ \sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5.3 - 3)$$

Defining

$$\bar{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k} \tag{5.3 - 4a}$$

$$\bar{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k} \tag{5.3 - 4b}$$

we get

$$\frac{\partial m}{\partial \theta_g} = \sum_{i=1}^{3}\sum_{j+1}^{3}\frac{\partial m}{\partial u_j}\frac{\partial u_j}{\partial w_i}\frac{\partial w_i}{\partial \theta_g} \tag{5.3 - 5}$$

where m is the measurement. We also have

$$\frac{\partial w_i}{\partial \theta_g} = \sum_{k=1}^{3}\frac{\partial R_{im}}{\partial \theta_g}u_k \tag{5.3 - 6}$$

and

$$\frac{\partial u_j}{\partial w_i} = R_{ji}{}^{-1} \tag{5.3 - 7}$$

Then

$$\frac{\partial m}{\partial \theta_g} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial m}{\partial u_j}R_{ji}^{-1}\frac{\partial R_{ik}}{\partial \theta_g}u_k \tag{5.3 - 8}$$

We can show that

$$\sum_{i=1}^{3}R_{ji}^{1}\frac{\partial R_{ik}}{\partial \theta_g} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.3 - 9}$$

Using (5.3-10), equation (5.3-9) takes the form:

$$\frac{\partial m}{\partial \theta_g} = \frac{\partial m}{\partial u_2}u_1 - \frac{\partial m}{\partial u_1}u_2 \tag{5.3 - 10}$$

Since

$$\frac{\partial \theta_g}{\partial UT1 = \dot{\theta}} \tag{5.3 - 11}$$

where $\dot{\theta}$ is the rotation rate of the earth, we get

$$\frac{\partial m}{\partial (A1 - UT1)} = -\frac{\partial m}{\partial UT1} = -\frac{\partial m}{\partial \theta_g}\frac{\partial \theta_g}{\partial UT1} \qquad = -\dot{\theta}\left[\tfrac{\partial m}{\partial u_2}u_1 - \tfrac{\partial m}{\partial u_1}u_2\right] \tag{5.3 - 12}$$

5.4 POLAR MOTION

Consider the point P which is defined by the intersection of the Earth's axis of rotation at some time t with the surface of the Earth. At some time $t + \Delta t$,the intersection will be at some point P' which is different than P. Thus the axis of rotation appears to be moving relative to a fixed position on the Earth; hence the term "motion of the pole." Let us establish a rectangular coordinate system centered at a point F fixed on the surface of the Earth with F near the point P around 1900, and take measurements of the rectangular coordinates of the point P during the period 1900.0 - 1906.0. It is observed that the point P moves in 'roughly circular motion in this coordinate system with two distinct periods, one period of approximately 12 months and one period of 14 months. We define the mean position of P during this period to be the point P_0 ,the mean pole of 1900.0 - 1906.0. The average is taken over a six year period in order to average out both the 12 month period and the 14 month period simultaneously (since 6 * 12 months = 72 months and 72 / 14 = 5 periods approximately of the 14 month term). The radius of this observed circle varies between 15 - 35 feet. In addition to the periodic motion of P about P_0 , by taking six year means of P in the years after 1900 - 1906, called P_m there is seen to be a secular motion of the mean position of the pole away from its original mean position P_0 in the years 1900 - 1906 at the rate of approximately 0".0032 per year in the direction of the meridian 60 W, and a Iibration motion of a period of approximately 24 years with a coefficient of about 0".022 The short periodic motions over a period of six years average about 0".2 - 0".3 .

5.4.1 Effect on the Position of a Station

This motion of the pole means that the observing stations are moving with respect to our "Earth- fixed" coordinate system used in GEODYN. The station positions must be corrected for this effect. The position of the instantaneous or true pole is computed by linear interpolation in a table of observed values for the true pole relative to the mean pole of 1900 - 1905. The table increment is 5 days; the current range of data is from December I, 1960 to June 1, 1972. The user should be aware of the fact that this table is expanded as new information becomes available. If the requested time is not in the range of the table, the value for the closest time is used. The data in the table is in the form of the coordinates of the true pole relative to the mean pole measured in seconds of arc. This data was obtained from "Circular 0" which is published by BIH.The appropriate coordinate system and rotation are illustrated in Figures 5.4-1 and 5.4-2. Consider the station vector $\bar{X}$ in a system attached to the Earth of the mean pole and the same vector $\bar{Y}$in the "Earth-fixed" system of GEODYN. The transformation between $\bar{Y}$ and $\bar{X}$ consists of a rotation of x about the X_2 axis and a rotation of j about the X_1 axis; that is

$$\bar{Y} = R_1(y)R_2(x)\bar{X} \qquad (5.4 - 1)$$

$$\bar{Y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos y & \sin y \\ 0 & -\sin y & \cos y \end{bmatrix} \begin{bmatrix} \cos x & 0 & -\sin x \\ 0 & 1 & 0 \\ \sin x & 0 & \cos x \end{bmatrix} \bar{X}$$

Because x and y are small angles, their cosines are set to 1 and their sines equal to their values in radians. Consequently,

$$\bar{Y} = \begin{bmatrix} 1 & 0 & -x \\ xy & 1 & y \\ x & -y & 1 \end{bmatrix} \bar{X} \qquad (5.4 - 2)$$

Figure 5.4-1 Coordinates of the Instantaneous Axis of Rotation
P_A = Center of Coordinate System
= Adopted Mean Pole
X_1 = Direction of 1^{st} Principal Axis (along meridian directed to Greenwich)
X_2 = Direction of 2^{nd} Principal Axis (along 90°West meridian)
P_T = Instantaneous Axis of Rotation
x, y = Coordinate of P_T relative to P_A measured in seconds of arc

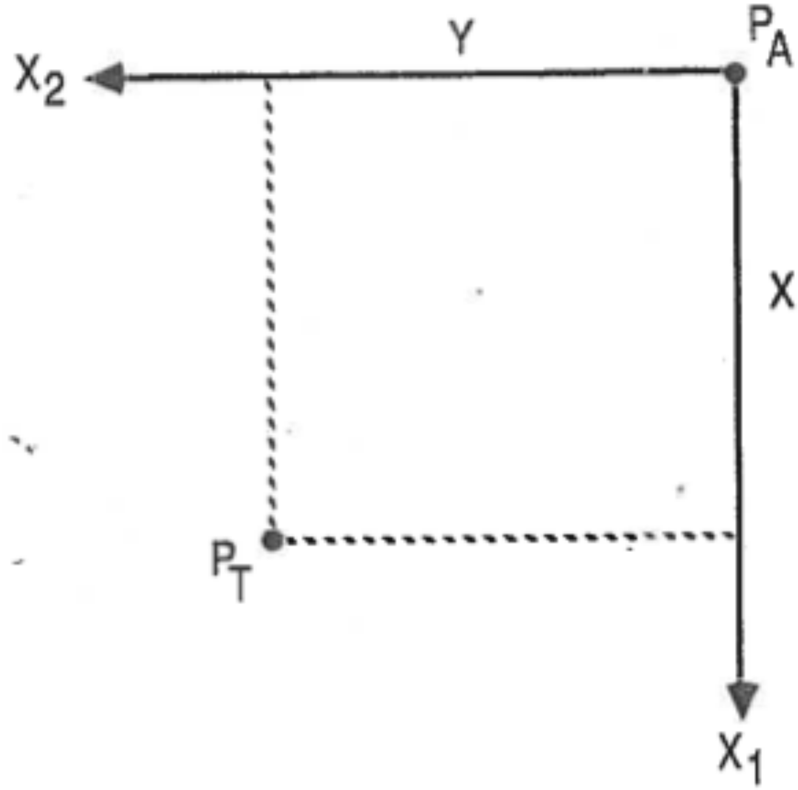
Figure 5.4-2 Rotation of Two Coordinate Systems: from Adopted Mean Pole System to Instantaneous Pole System
x, y = Rectangular Coordinates of P_T relative to P_A
X_1X_2 Plane =Mean Adopted Equator Defined by Direction of Adopted Pole P_A
Y_1Y_2 Plane = Instantaneous Equator Defined by Direction of Instantaneous Pole P_T

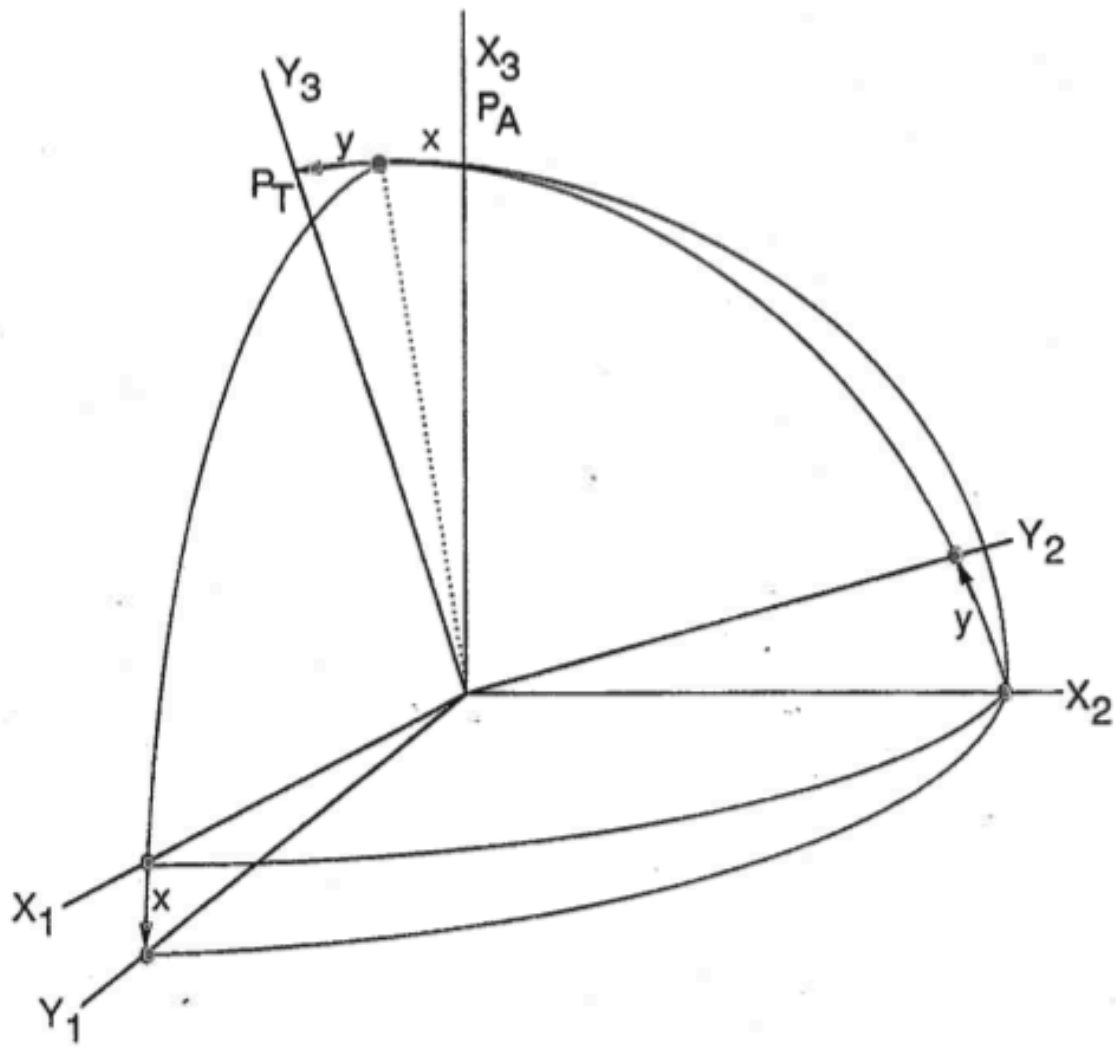
5.4.2 Partial Derivatives

The coordinate rotation is defined as

$$\bar{u} = R_1(y)R_2(x)\bar{w} \qquad (5.4 - 3)$$

where
$\bar{w}$ = station vector in a system attached to the Earth of the mean pole.
$\bar{u}$ = station vector in a system attached to the Earth of the true pole.
$R_1(y)$ = matrix of rotation about the Xl axis.
$R_2(x)$ = matrix of rotation about the X2 axis.


Y
P_A
X_2
X
P_T
X_1



X3
PA
Y3
y
x
PT
Y2
y
X2
X1
x
Y1

The rotation matrices are:

$$R_1(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos y & \sin y \\ 0 & -\sin y & cosy \end{bmatrix}$$

$$R_2(x) = \begin{bmatrix} \cos x & 0 & -\sin x \\ 0 & 1 & 0 \\ \sin x & 0 & \cos x \end{bmatrix}$$

Defining

$$\bar{u} = u_i\hat{i} + u_2\hat{j} + u_3\hat{k} \tag{5.4 - 4}$$

$$\bar{w} = w_i\hat{i} + w_2\hat{j} + w_3\hat{k} \tag{5.4 - 5}$$

$$\tag{5.4 - 6}$$

and performing the matrix multiplications.

$$u_1 = w_1 \cos x - w_3 \sin x$$

$$u_2 = w_1 \sin x \sin y + w_2 \cos y + w_3 \cos x \sin y \tag{5.4 - 7}$$

$$u_3 = w_1 \sin x \cos y - w_2 \sin y + w_3 \cos x \cos y$$

The fundamental quantities required for the estimation of polar motion parameters are:

$$\frac{\partial m}{\partial x} and \frac{\partial m}{\partial y}$$

where m is the satellite observation.
Using the chain rule

$$\frac{\partial m}{\partial x} = \frac{\partial m}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial m}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial m}{\partial u_3}\frac{\partial u_3}{\partial x}$$

$$\frac{\partial m}{\partial y} = \frac{\partial m}{\partial u_1}\frac{\partial u_1}{\partial y} + \frac{\partial m}{\partial u_2}\frac{\partial u_2}{\partial y} + \frac{\partial m}{\partial u_3}\frac{\partial u_3}{\partial y} \tag{5.4 - 8}$$

The partial derivatives of the satellite observation with respect to the true station coordinates are currently available in GEODYN. The partial derivatives of the station coordinates with respect to the polar motion parameters arc:

$$\frac{\partial u_1}{\partial x} = -w_1 \sin x - w_3 \cos x$$

$$\frac{\partial u_1}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial x} = w_1 \cos x siny - w_3 \sin x \sin y \qquad (5.4 - 9)$$

$$\frac{\partial u_2}{\partial y} = w_1 \sin x \cos y - w_2 \sin y + w_3 \cos x \cos y$$

$$\frac{\partial u_3}{\partial x} = w_1 \cos x \cos y - w_3 \sin x \cos y$$

$$\frac{\partial u_3}{\partial y} = -w_1 \sin x \sin y - w_2 \cos y - w_3 \cos x \sin y$$

Since the angles x and y are small, the following approximations may be made:

$$\sin x = x \cos x = 1$$

$$\sin y = y \cos y = 1 \qquad (5.4 - 10)$$

Substituting equations (5.4-9) into equations (5.4-8)

$$\frac{\partial u_1}{\partial x} = -w_1 x - w_3$$

$$\frac{\partial u_1}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial x} = w_1 y - w_3 x y \qquad (5.4 - 11)$$

$$\frac{\partial u_2}{\partial y} = w_1 x - w_2 y + w_3$$

$$\frac{\partial u_3}{\partial x} = w_1 - w_3 x$$

$$\frac{\partial u_3}{\partial y} = -w_1 x y - w_2 - w_3 y$$

GEODYN allows the user the option of co straining the estimation of the pole to a specified longitude. When a longitudinal constraint is requested, a new coordinates system X', Y' is defined so that the X' - coordinate points along the constant meridian. The Y' - coordinate points along the meridian 270° east of the constraint meridian. The a priori uncertainty in the X - position of the pole, as obtained from the input data stream is interpreted as the a priori uncertainty of the X' - position of the pole. The Y' uncertainty is set internally in the program to be very small. The covariances arc set to zero. By this procedure, the program constructs a diagonal variance - covariance matrix:

$$V = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & 10^{-12} \end{bmatrix}$$

The variance - covariance matrix in the(X', Y') system is then rotated to form the variance - covariance matrix in the (X, Y) coordinate terms. In this manner, the longitude constraint is equivalent to the specification of an a priori correlation coefficient.

5.5 SOLID EARTH TIDES

5.5.1 Station Displacement Due to Luni-Solar Attraction

The station displacement due to tidal potential is computed using Wahr's theory, where only the second degree tides are needed for a centimeter level precision ([5 - 5] and [5 - 6]). The formulation uses frequency independent Love and Shida numbers and a derivation of the tidal potential in the time domain. The formula for the vector displacement as developed by Diamante and Williamson [5 - 2] is the following:

$$\Delta\bar{r} = \sum_{j=2}^{3} \left[\frac{GM_j}{GM_\oplus} \frac{r^4}{R_j^3} \right] \left\{ [3l_2(\hat{R}_j \cdot \hat{r})]\hat{R}_j + [3(\frac{h_2}{2} - l_2)(\hat{R}_j \cdot \hat{r})^2 - \frac{h_2}{2}]\hat{r} \right\} \qquad (5.5 - 1)$$

GM_j = gravitational parameter for the Moon U = 2) or the Sun (j = 3),
$GM_\oplus$ = gravitational parameter for the Earth,
$\hat{R}_j, R_j$ = unit vector from the geocenter to Moon or Sun and the magnitude of that vector,
$\hat{r}, r$ = unit vector from the geocenter to the station and the magnitude of that vector,
h_2 = nominal second degree Love number,
l_2= nominal Shida number.

If nominal values for l_2 and h_2 are used (i.e. 0.609 and 0.0852 respectively) only one term in the above equation needs to be corrected at the 5mm level. This is the k_1 frequency, where from Wahr's theory h_{k_1} is 0.5203. Then the radial displacement is a periodic change in station height given by:

$$\delta h_{STA} = \delta h_{k_1} H_{k_1} \left(-\sqrt{\frac{5}{24\pi}} \right) 3 \sin\phi \cos\phi \sin(\theta_{k_1} + \lambda), \qquad (5.5 - 2)$$

where
δh_{k_1} = h_{k_1} (Wahr) - h_2 (Nominal) = -0.0887,
H_{k_1} = amplitude of K_1 term (165.555) in the harmonic expansion of the tide generating potential = 0.36878 m,
ϕ = geocentric latitude of station,
λ = east longitude of station,
θ_{K_1} = K_1 tide argument = τ + s = $\theta_g + \pi$,
θ_g = Greenwich hour angle.
In GEODYN II the displacement is computed in cartesian coordinates ΔX, ΔY I and ΔZ. Given the cartesian components ($\bar{X}_j$, $\bar{Y}_j$, $\bar{Z}_j$ and ($\bar{X}_{STA}$, $\bar{Y}_{STA}$, $\bar{Z}_{STA}$) of the unit vectors

$\hat{R}_j$ and $\bar{r}$; respectively, the total correction for the solid tide is given by:

$$\Delta X = \sum_{j=2}^{3}\left[\frac{GM_j}{GM_\oplus}\frac{r^4}{R_j^3}\right]\left\{[3l_2(\hat{R}_j\cdot\hat{r})]\hat{X}_j + [3(\frac{h_2}{2}-l_2)(\hat{R}_j\cdot\hat{r})^2 - \frac{h_2}{2}]\bar{X}_{STA}\right\}$$
$$+\,\delta h_{k_1}H_{k_1}\left(-\sqrt{\frac{5}{24\pi}}\right)3\sin\phi\cos\phi\sin(\theta_{k_1}+\lambda)\bar{X}_{STA} \quad (5.5\text{ - }3)$$

The expressions for ΔY and ΔZ are derived similarly, by substituting $\bar{Y}_j$, $\bar{Y}_{STA}$ and $\bar{Z}_j$, $\bar{Z}_{STA}$ for $\bar{X}_j$, $\bar{X}_{STA}$, respectively.

5.5.2 Station Displacement Due to Pole Tide

To describe the deformation undergone by the solid earth in response to polar wobble, the concept outlined by Vanicek, p. 603 [5 - 7] was used for the present formulation. The centripetal potential is expressed as a function of time t by:

$$W_c(t) = \frac{1}{2}\omega^2 p^2 = \frac{1}{2}\omega^2(X^2+Y^2) = \frac{1}{2}\omega^2 r^2\cos^2\phi \quad (5.5\text{ - }4)$$

where
ω = the mean angular velocity of rotation bf the Earth.

Differentiation with respect to time yields in a first order of approximation:

$$\frac{\partial W_c}{\partial\phi} \simeq -\frac{1}{2}\omega^2 r^2\sin 2\phi \quad (5.5\text{ - }5)$$

After defining the latitude change (due to polar motion) , the perturbation of the potential is given by:

$$W_p \simeq -\frac{1}{2}\omega^2 r^2\sin 2\phi\delta\phi \quad (5.5\text{ - }6)$$

where

$$\delta\phi = \phi - \phi_0 = x_p\cos\lambda_0 - y_p\sin\lambda_0 \quad (5.5\text{ - }7)$$

The uplift of the solid earth due to W_p is approximated by:

$$\delta H = u_h = \frac{W_p}{g} \quad (5.5\text{ - }8)$$

The effect of the potential change W_p in the longitudinal and latitudinal directions (in metric units) is found from:

$$u_\phi = \frac{1}{g}\frac{\partial W_g}{\partial \phi} \tag{5.5 - 9}$$

and

$$u_\lambda = \frac{1}{g\cos\phi}\frac{\partial W_p}{\partial \lambda} \tag{5.5 - 10}$$

Equations (5.5-8) to (5.5-10) pertain to a rigid earth. Considering an elastic earth by introducing the Love and Shida numbers, h_2 and h_1, for radial and horizontal displacement, respectively we have:

$$u_h = -h_2\frac{\omega^2 r^2}{2g}\sin 2\phi\delta\phi \tag{5.5 - 11}$$

$$u_\phi = -l_2\frac{\omega^2 r^2}{g}\cos 2\phi\delta\phi \tag{5.5 - 12}$$

$$u_\lambda = l_2\frac{\omega^2 r^2}{g}\sin\phi(x_p\sin\lambda + y_p\cos\lambda) \tag{5.5 - 13}$$

For the computations of the above displacements the values of r and g are taken as constants; namely the earth mean radius r = R_m = 6371 km and $g = g_{ave}$= $9.81\frac{m}{s^2}$.

Finally, the displacements u_ϕ, u_λ, and u_h, referring to the local coordinate system u, $-v$ and w, are rotated into the Cartesian coordinate system for GEODYN II.

$$\begin{bmatrix} u_\phi \\ -u_\lambda \\ u_h \end{bmatrix} = \begin{bmatrix} u \\ -v \\ w \end{bmatrix} = R_2(-(90^\circ - \phi))R_3(-(180^\circ - \lambda))\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{5.5 - 14}$$

which is inverted easily due to the orthogonality of the rotations involved.

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} -\sin\phi\cos\lambda & \sin\lambda & \cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & \cos\phi\sin\lambda \\ \cos\phi & 0 & \sin\phi \end{bmatrix}\begin{bmatrix} u_\phi \\ u_\lambda \\ u_h \end{bmatrix} \tag{5.5 - 15}$$

For an alternative derivation, see the IERS standards [5 - 6].

5.5.3 Partial Derivatives for h2 and l2

The partial derivative quantities required for the estimation of the Love and Shida numbers h_2 and l_2 are:

$\frac{\partial m}{\partial h_2}$ and $\frac{\partial m}{\partial l_2}$ where m is a given measurement.

If $\bar{r}$ is the geocentric station vector, using the chain rule,

$$\frac{\partial m}{\partial h_2} = \frac{\partial m}{\partial X}\frac{\partial X}{\partial h_2} + \frac{\partial m}{\partial Y}\frac{\partial Y}{\partial h_2} + \frac{\partial m}{\partial Z}\frac{\partial Z}{\partial h_2} \qquad (5.5 - 16)$$

$$\frac{\partial m}{\partial l_2} = \frac{\partial m}{\partial X}\frac{\partial X}{\partial l_2} + \frac{\partial m}{\partial Y}\frac{\partial Y}{\partial l_2} + \frac{\partial m}{\partial Z}\frac{\partial Z}{\partial l_2} \qquad (5.5 - 17)$$

GEODYN II computes the partial derivatives in two parts. The first part is due to the solid tide [eq. (5.5-3)] and it is computed from:

$$\left[\frac{\partial X}{\partial h_2}\right]_1 = \sum_{j=2}^{3}\left[\frac{GM_j}{GM_\oplus}\frac{r^4}{R_j^3}\right]\left[\frac{3}{2r}(\hat{R}_j \cdot \hat{r})^2] - \frac{1}{3}\right]X + \\ + \frac{H_{k_1}}{r} 3\sqrt{\frac{5}{24\pi}}\cos\phi\sin\phi\sin(\theta_{k_1} + \lambda)X \qquad (5.5 - 18)$$

$$\left[\frac{\partial X}{\partial l_2}\right]_1 = 3\sum_{j=2}^{3}\frac{GM_j}{GM_\oplus}\frac{r^4}{R_j^3}\left[3(\hat{r} \cdot \hat{R}_j)\bar{X}_j - \frac{1}{r}(\hat{R}_j \cdot \hat{r})^2 X\right] \qquad (5.5 - 19)$$

Similarly the derivatives of Y and Z are computed.

The second part is added to the above partials when the application of the correction due to pole tide is requested (through the TIDES option card).

$$\left[\frac{\partial X}{\partial h_2}\right]_2 = -\frac{\omega^2 r^2}{g}\delta\phi\cos^2\phi\sin\phi\cos\lambda \tag{5.5 - 20}$$

$$\left[\frac{\partial Y}{\partial h_2}\right]_2 = -\frac{\omega^2 r^2}{g}\delta\phi\cos^2\phi\sin\phi\sin\lambda \tag{5.5 - 21}$$

$$\left[\frac{\partial Z}{\partial h_2}\right]_2 = -\frac{\omega^2 r^2}{g}\delta\phi\cos\phi\sin^2\phi \tag{5.5 - 22}$$

$$\left[\frac{\partial X}{\partial l_2}\right]_2 = \frac{\omega^2 r^2}{g}[(1-2\sin^2\phi)\sin\phi\cos\lambda\delta\phi - (x_p\sin\lambda + y_p\cos\lambda)\sin\phi\sin\lambda] \tag{5.5 - 23}$$

$$\left[\frac{\partial Y}{\partial l_2}\right]_2 = \frac{\omega^2 r^2}{g}[(1-2\sin^2\phi)\delta\phi\sin\phi\sin\lambda + (x_p\sin\lambda + y_p\cos\lambda)\cos\lambda\sin\phi] \tag{5.5 - 24}$$

$$\left[\frac{\partial Z}{\partial l_2}\right]_2 = -\frac{\omega^2 r^2}{g}\delta\phi(1-2\sin^2\phi)\cos\phi \tag{5.5 - 25}$$

The default values are h_2= 0.6 and l_2 = 0.075 although the nominal values of 0.609 and 0.0852 respectively can be used by means of the H2LOVE and L2LOVE option cards.

5.6 THE UPLIFT DUE TO OCEAN TIDES

5.6.1 The Hendershott Tidal Model

The theoretical equations governing ocean tidal motion are the Laplace Tidal Equations:

$$\frac{\partial u}{\partial(t)} - (2\Omega\sin\phi')v = \frac{-g}{a\cos\phi'}\frac{\partial}{\partial\lambda}(\zeta - \frac{\Gamma}{g}) + \frac{F_\lambda}{\rho D} \tag{5.6 - 1}$$

$$\frac{\partial v}{\partial(t)} + (2\Omega\sin\phi')u = \frac{-g}{a}\frac{\partial}{\partial\phi'}(\zeta - \frac{\Gamma}{g}) + \frac{F_{\phi'}}{\rho D} \tag{5.6 - 2}$$

$$\frac{\partial}{\partial(t)}(\zeta - \delta) - \frac{1}{a\cos\phi'}\left[\frac{\partial}{\partial\lambda}(uD) + \frac{\partial}{\partial\phi'}(vD\cos\phi')\right] = 0 \tag{5.6 - 3}$$

where λ and ϕ' are the longitude and geocentric latitude, u and v corresponding velocity components, a is the earth's radius, Ω is the earth's angular rotation rate and g is the

acceleration of gravity. F_λ and F'_ϕ are the bottom stresses and D is the depth of the ocean. ρ is the ocean water density, ζ s the ocean tide amplitude and δ is the solid earth tide amplitude.
The total tide generating potential, Γ, is composed of

1. The astronomical contribution, U, from the moon and sun
2. A component due to the solid earth yielding under the influence of the astronomical body force.
3. A component due to the potential of the layer of ocean water of thickness ζ_0 .
4. A component due to the solid earth yielding under the influence of the ocean load of thickness ζ_0.

Solutions to the Laplace Tidal Equations are exceedingly difficult to obtain for any geophysically meaningful set of tidal parameters and boundary conditions. The solution due to Hendershott (1972) is representative of the current state of the art and provides one of the few ocean tidal models of any interest for satellite altimetry applications. In this solution, Hendershott has represented the astronomical tide generating potential by the dominant second order term, U_2, in the spherical harmonic expansion. The effects of the ocean self-attraction and the ocean loading effect are neglected (Hendershott, 1972). The solution only represents the dominant lunar semi- diurnal tide (M_2) with lunar declination terms neglected. A harmonic analysis of the M_2 tide measurements along the coastal regions of the continents was employed as boundary conditions and in this sense, dissipation effects were included in the regions of the continental shelves and inIand seas, but are otherwise neglected in the Laplace Tidal Equations. The computational boundaries were consequently the seaward edges of the continental shelves and marginal seas. Within the computational region D is assumed constant.
The Hendershott solution is represented as a spherical harmonic expansion (in complex form):

$$(\zeta_0)_{M_2} = Real\left[\sum_{n=0}^{25}\sum_{m=0}^{n} P_n^{-m}(\sin\phi')(\bar{a}_{nm}\cos m\lambda + \bar{b}_{nm}\sin m\lambda)exp(-i\omega_{M_2}t)\right](cm.) \quad (5.6 - 4)$$

where

$$\bar{a}_n m = a_{nm} + ic_{nm}$$
$$\bar{b}_n m = a_{nm} + id_{nm} \quad (5.6 - 5)$$

(in real form):

$$(\zeta_0)_{M_2} = \sum_{n=0}^{25}\sum_{m=0}^{n} P_n^{-m}(\sin\phi')[(a_{nm}\cos\omega t + c_{nm}\sin\omega t) + (b_{nm}\cos\omega t + d_{nm}\sin\omega t)\sin m\lambda]$$

(5.6 - 6)

Letting $\mu = \sin\phi'$, the associated Legendre functions P_n^{-m} are defined by:

$$\bar{P}_n^m(\mu) = (-1)^m \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} \frac{(1-\mu^2)^{\frac{m}{2}}}{2^n n!} \frac{d^{m+n}}{d\mu^{m+n}} (\mu^2 - 1)^n \qquad (5.6 - 7)$$

The normalization employed is that introduced by Backus (1958):

$$\int_0^{2\pi} d\lambda \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (P_n^{-m}(\mu)^{\sin}_{\cos} m\lambda)^2 \cos\phi' d\phi' = \begin{cases} 1 & \text{m=0} \\ \frac{1}{2} & \text{m>0} \end{cases} \qquad (5.6 - 8)$$

The unnormalized associated Legendre Functions $P_n^m \mu$ have the property

$$\int_0^{2\pi} d\lambda \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (P_n^{-m}(\mu)^{\sin}_{\cos} m\lambda)^2 \cos\phi' d\phi' = \begin{cases} \frac{4\pi}{2n+1} & \text{m=0} \\ \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!} & \text{m>0} \end{cases} \qquad (5.6 - 9)$$

Therefore,

$$\bar{P}_n^m(\mu) = \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} P_n^m(\mu) \qquad (5.6 - 10)$$

The coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ may be transformed for application with unnormalized functions by the relation (McClure, 1973)

$$\begin{matrix} \bar{\bar{a}}_{nm} \\ \bar{\bar{b}}_{nm} \end{matrix} = \left[\frac{4\pi}{2n+1} \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} \begin{matrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{matrix} \qquad (5.6 - 11)$$

While Hendershott (1972) obtained coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ up to n = 25, he has only published results for $0 < n < 9$.

5.6.2 The NOAA Tide Model

To calculate the ocean tide displacement globally, GEODYN uses the Hendershott model as described in Section 5.6.1. However, for an area in the western North Atlantic from approximately -77° to -62° E longitude, and 27° to 35° N latitude, GEODYN calculates the tidal displacement by using an empirical model developed by the National Oceanic and Atmospheric Administration (NOAA). This model computes semi-daily and daily tides based on constants interpolated from three reference stations in this area of the ocean.

For this model the sea surface displacement h at a given time and location is given by

$$h = \sum_{i=1}^{8} f_1 A_i \cos(\sigma_i t - \zeta_i) \tag{5.6 - 12}$$

Here f_i, A_i, σ_i, and ζ_i are respectively the node factor, amplitude, frequency, and phase lag of the i^{th} tidal constituent, and t is the time relative to 0000 GMT on 1 March 1975. The frequencies of σ_i of the eight principal tidal constituents are taken from Schureman (1941). The node factors f_i are computed from cubic polynomials, as found in Schureman (1941), and given by

$$f_i = a_i + b_i u + c_i u^2 + d_i u^3 \tag{5.6 - 13}$$

where a_i, b_i, c_i, and d_i are coefficients, and $u = t - t_0$, where t_0 is the time from 0000 GMT 1 July 1975 to the start time of the model.
The amplitude A_i, and the phase lags ζ_i are computed from the complex harmonic constants $H_i = (H_i', H_i'')$,

$$A_i = (H_t'^2 + H_i''^2)^{\frac{1}{2}} \tag{5.6 - 14}$$

and

$$\zeta_i = \tan^{-1}(\frac{H_t''}{H_t'}) \tag{5.6 - 15}$$

The complex harmonic constants are computed at a given location by the linear polynomial:

$$H_i = (H_{i,1}X = (H_{i,2})Y + H_{i,3}$$

where $H_{i,1}$, $H_{i,2}$ and $H_{i,3}$ are coefficients and X and Y are the zonal and meridianal Mercator coordinates, corresponding to the latitude θ and east longitude λ of the location, measured westward as a negative quantity for 0° E, i.e.,

$$X = \pi\lambda \tag{5.6 - 16}$$

and

$$Y = ln\{\tan(45^\circ + \frac{\theta}{2})\}$$

The coefficients $H_{i,j}$n are found by fitting equation (5.6-16) to complex harmonic constants at three reference stations. A complete description of the NOAA tide tide model is given in Mofjeld (1975).

5.7 TECTONIC PLATE MOTION

One of the factors which can effect Earth rotation results, is the motion of the tectonic plates which make up the earth's surface. As the plates move, fixed coordinates of the observing stations will become inconsistent with each other. The rates of relative motions for some regular observing sites are believed to be 5 em per year or larger. In order to reduce inconsistencies in the station coordinates, a model for plate motions based on the relative plate motion model RM-2 of Minster and Jordan (1978) is recommended in the Merit Standard (Dee 85). From the RM-2 model, Minster and Jordan (1978) derive four different absolute plate motion models. The two which have been discussed most widely are AMO-2, which has a zero net rotation of the Earth's surface, and AMI-2 which minimizes the motion of a set of hot spots. Both of these models can be applied in GEODYN-II by the users request in the input file.

The effect on the stations position in an earth fixed Cartesian coordinate system, is given by:

$$\begin{aligned} x &= x_o + v_x \Delta t \\ y &= y_o + v_y \Delta t \\ z &= z_o + v_z \Delta t \end{aligned} \qquad (5.7 - 1)$$

where:
$[x\ , y, z]$ are the updated station coordinates
$[x_o, y_o, z_o]$ are the a-priori station coordinates.
$[v_x, v_y, v_z]$ are the components of the plate velocities in an earth fixed Cartesian coordinate system, and
Δt is the time difference between the plate motion reference time and the time of the particular observation.

The vector of velocities $[\ v_x, v_y, v_z]$ is given by the following equation:

$$\bar{v} = R^{-1}(\bar{\omega} \times \bar{r}) \qquad (5.7 - 2)$$

where:
$\bar{v} = [\ v_x, v_y, v_z]$ the velocity vector in the earth centered fixed Cartesian coordinate system.
R = rotation matrix needed to convert the vectors from earth centered Cartesian to local rotation system (plate rotation).

$\bar{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$ to rotational velocity vector, assuming only z axis rotation, provided by the model.

$$\bar{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R = \begin{bmatrix} -\sin\lambda & \cos\lambda\sin\phi & \cos\lambda\cos\phi \\ -\cos\lambda & -\sin\lambda\sin\phi & -\sin\lambda\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix}$$

where ϕ and λ are the latitude and longitude of the the origin of the plate rotation, provided by the model.

The final equations in GEODYN II have the following form:

$$\begin{aligned} v_x &= \sin\lambda\omega y - \cos\lambda\omega x \\ v_y &= -\cos\lambda\sin\phi\omega y - \sin\lambda\sin\phi\omega x \\ v_z &= -\cos\lambda\cos\phi\omega y - \sin\lambda\cos\phi\omega x \end{aligned} \qquad (5.7 - 3)$$

5.8 OCEAN LOADING

The redistribution of the ocean mass due to tides causes a mass disturbance of the solid Earth, especially in the coastal areas. This effect is called ocean loading and it has the same periodicity as the ocean tides. It is expressed as a vertical displacement of the stations, which is added to the radial component of the station coordinates. Project Merit Standards (USNO Circular No. 167) and lERS Technical Note 3 [p.43] give for various sites phases and amplitudes of the displacement in the radial direction due to the eleven main tides (M_2, S_2, K_2, N_2, O_1, K_1, P_1, Q_1, M_f, M_m and S_{sa}), using models generated by Schwiderski (1978). The Merit model for height displacement due to the ith constituent at time t is:

$$h(i) = amp(i) \times \cos(arg(i,t) - phase(i)$$

The total tidal displacement is found by summing over all the constituents. A negative $h(i)$ means that the surface at the predicted point has been lowered. GEODYN uses the following formula for the height displacement.

$$h(i) = A(i) \times \cos(arg(i, t + B(i) \times \sin(arg(i,t))$$

Considering the Merit convention of phase and amplitude (with the phase being subtracted) GEODYN's A and B coefficients relate to the Merit phase and amplitude as follows:

$$A(i) = amp(i) \times \cos(phase(i))$$

$B(i) = amp(i) \times \sin(phase(i))$

The argument $arg(i, t)$ is a linear combination of certain astronomical angles which relate to the positions of the sun and the moon. This linear combination can be deduced from the input Doodson number. However, in order to be consistent with the Merit standards, certain Doodson numbers which correspond to diurnal frequencies have a phase added to the angle directly implied by the Doodson number. The following is a list of the Doodson numbers for which the above argument is true,

Darwinian Symbol	Doodson Number	Phase(degrees)
K_1	165555	270
O_1	145555	90
P_1	163555	90
Q_1	135655	90

6 MEASUREMENT MODELING AND RELATED DERIVATIVES

The observations in GEODYN are geocentric in nature. The computed values for the observations are obtained by applying these geometric relationships to the computed values for the relative positions and velocities of the satellite and the observer at the desired time. In addition to the geometric relationships, GEODYN allows for a timing bias and for a constant bias to be associated with a measurement type from a given station. Both of these biases are optional. The measurement model for GEODYN is therefore:

$$C_{t+\Delta t} = f_1(\bar{r}, \dot{\bar{r}}, \bar{r}_{ob} + b + f_t(\bar{r}, \dot{\bar{r}}, \bar{r}_{ob})\Delta t \qquad (6.0 - 1)$$

where
$C_{t+\Delta t}$ is the computed equivalent of the observation taken at time $t + \Delta t$,
$\bar{r}$ is the Earth-fixed position vector of the satellite,
$\bar{r}_{ob}$ is the Earth-fixed position vector of the station,
$f_t(\bar{r}, \dot{\bar{r}}, \bar{r}_{ob})$ is the geometric relationship defined by the particular observation type at time t,
b is a constant bias on the measurement, and
Δt is the timing bias associated with the measurement.

The functional dependence of f_t was explicitly stated for the general case. Many of the measurements are functions only of the position vectors and are hence not functions of the satellite velocity vector $\dot{\bar{r}}$, We will hereafter refer to without the explicit functional dependence for notational convenience. As was indicated earlier in Section 2.2, we require the partial derivatives of the computed values for the measurements with respect to the parameters being determined (see also Section 10.1). These parameters are:

- The true of date position and velocity of the satellite at epoch. These correspond to the inertial position and velocity which are the initial conditions for the equations of motion.

- force model parameters,

- the Earth-fixed station positions,

- measurement biases.

These parameters are implicitly divided into a set $\bar{\alpha}$ which are **not** concerned with the dynamics of satellite motion. and a set $\bar{\beta}$ which are.

The partial derivatives associated with the parameters $\bar{\alpha}$; i.e., station position and measurement biases are computed directly at the given observation times. The partial derivatives with respect to the parameters $\bar{\beta}$; i.e., the epoch position and velocity and the force model parameters, must be determined according to the chain rule:

$$\frac{\partial C_{t+\Delta t}}{\partial \bar{\beta}} = \frac{\partial C_{t+\Delta t}}{\partial \bar{x}_t} \frac{\partial \bar{x}_t}{\partial \bar{\beta}} \tag{6.0 - 2}$$

where
$\bar{x}_t$ is the vector which describes the satellite position and velocity in true of date coordinates.

The partial derivatives $\frac{\partial C_{t+\Delta t}}{\partial \bar{x}_t}$ are computed directly at the given observation times, but the partial derivatives $\frac{\partial \bar{x}_t}{\partial \bar{\beta}}$ may not be so obtained.

These latter relate the true of date position and velocity of the satellite at the given time to the parameters at epoch through the satellite dynamics.

The partial derivatives $\frac{\partial \bar{x}_t}{\partial \bar{\beta}}$ are called the variational partials and are obtained by direct numerical integration of the variational equations. As will be shown in Section 8.2, these equations are analogous to the equations of motion.

Let us first consider the partial derivatives of the computed values associated with the parameters in $\bar{\beta}$. We have

$$\frac{\partial C_{t+\Delta t}}{\partial \bar{\beta}} = \frac{\partial f_t}{\partial \bar{x}_t}\frac{\partial \bar{x}_t}{\partial \bar{\beta}} \tag{6.0 - 3}$$

Note that we have dropped the partial derivative with respect to $\bar{\beta}$ of the differential product $f_t \Delta t$, This is because we use a first order Taylor series approximation in our error model and hence higher order terms are. assumed negligible. This linearization is also completely consistent with the linearization assumptions made in the solution to the estimation equations (Section 10.1).

The partial derivatives $\frac{\partial f_t}{\partial \bar{x}_t}$ are computed by transforming the partial derivatives $\frac{\partial f_t}{\partial \bar{r}}$ and $\frac{\partial f_t}{\partial \dot{\bar{t}}}$ from the Earth-fixed system to the true of date system (see Section 3.4). These last are the partial derivatives of the geometric relationships given later in this section (6.2).

In summary, the partial derivatives required for computing the $\frac{\partial C_{t+\Delta t}}{\partial \bar{\beta}}$, the partial derivatives of the computed value for a given measurement, are the variational partials and the Earth-fixed geometric partial derivatives.

The partial derivatives of the computed values with respect to the station positions are simply related to the partial derivatives with respect to the satellite position at time t:

$$\frac{\partial C_{t+\Delta t}}{\partial \bar{r_{ob}}} = \frac{\partial f_t}{\partial \bar{r_{ob}}} = -\frac{\partial f_t}{\partial \bar{r}} \tag{6.0 - 4}$$

where $\bar{r}$ is of course the satellite position vector in Earth-fixed coordinates. This simple relationship is a direct result of the symmetry in position coordinates. The function f is a geometric function of the relative position; i.e., the differences in position coordinates which will be the same in any coordinate system.

The partial derivatives with respect to the biases are obvious:

$$\frac{\partial C_{t+\Delta t}}{\partial b} = 1 \tag{6.0 - 5}$$

$$\frac{\partial C_{t+\Delta t}}{\partial (\Delta t)} = f_t \tag{6.0 - 6}$$

In the remainder of this section, we will be concerned with the calculation of the geometric function f_t and its derivatives. These derivatives have been shown above to be the partial derivatives with respect to satellite position and velocity at time t and the time rate of change of the function, f_t.

The data preprocessing also requires some use of these formulas for computing measurement equivalents.

6.1 THE GEOMETRIC RELATIONSHIPS

The basic types of observation in GEODYN are:

- range
- range rate
- interplanetary doppler model
- altimeter height.
- right ascension and declination
- 1 and m direction cosines
- X and Y angles
- azimuth and elevation

The geometric relationship which corresponds to each of these observations is presented below. It should be noted that in addition to the Earth-fixed or inertial coordinate systems, some of these utilize topocentric coordinate systems. These last are presented in Section 5.2.

Range

Consider the station satellite vector:

$$\bar{\rho} = \bar{r} - \bar{r_{ob}} \qquad (6.1 - 1)$$

where
$\bar{r}$ is the satellite position vector (x, y, z) in the geocentric Earth-fixed system, and
$\bar{r_{ob}}$ is the station vector in the same system.

The magnitude of this vector, ρ, is the (slant) range, which is one of the measurements.

Range rate

The time rate of change of this vector $\bar{\rho}$ is

$$\dot{\bar{\rho}} = \dot{\bar{r}} \tag{6.1 - 2}$$

as the velocity of the observer in the Earth-fixed system is zero. Let us consider that

$$\bar{\rho} = \rho\hat{u} \tag{6.1 - 3}$$

where
$\hat{u}$ is the unit vector in the direction of ρ Thus we have

$$\dot{\bar{\rho}} = \dot{\rho}\hat{u} + \rho\hat{u} \tag{6.1 - 4}$$

The quantity $\dot{\rho}$ in the above equation is the computed value for the range rate and is determined by

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{r}} \tag{6.1 - 5}$$

Interplanetary Doppler Model

The usual formulation of range rate computations involves the simple differencing of two subsequent ranges divided by the counting interval. This differencing does not present any significant numerical problems for an Earth orbiting satellite but, because of the size of the ranges involved, it does for the interplanetary case. Using 64-bit arithmetic gives us approximately 14 significant digits. For an interplanetary run, where the ranges are on the order of l0e12 meters and greater, precision is limited to 1 cm in range and 2.0e-4 m/s in range-rate for a 60 second counting interval. This is one to two orders of magnitude above the Mars Observer mission requirements. A technique has been developed which computes the necessary difference without the induced numerical error.

This new model, mainly implemented in subroutine INTPRR, is based upon the fact that the range difference can be expanded into a multidimensional Taylor series. The range at time 1 is

$$r_1 = SQRT[(X_E + X_{ST}) - (X_P + X_S)]^2 + [(Y_E + Y_{ST}) - (Y_P + Y_S)]^2 + [(Z_E + Z_{ST}) - (Z_P + Z_S)]^2$$

and the range at time 2 is

$$\begin{aligned} r_2 = SQRT[&(X_E + \Delta X_E + X_{ST} + \Delta X_{ST}) - (X_P + \Delta X_P + X_S + \Delta X_S)]^2 + \\ &[(Y_E + \Delta Y_E + Y_{ST} + \Delta Y_{ST}) - (Y_P + \Delta Y_P + Y_S + \Delta Y_S)]^2 + \\ &[(Z_E + \Delta Z_E + Z_{ST} + \Delta Z_{ST}) - (Z_P + \Delta Z_P + Z_S + \Delta Z_S)]^2 \end{aligned}$$

where E is the barycentric Earth Position, P is the barycentric central body position, S is the planet-centered satellite positron (barycentric frame), and ST is the earth-centered station coordinates (barycentric frame). The range at time 2 can be expressed in terms of the range at time 1 as follows:

$$r_2 = SQRT\Big[r_1^2 +$$
$$2(X_E + X_{ST} - X_P - X_S)(\Delta X_E + \Delta X_{ST} - \Delta X_P - \Delta X_S) + (\Delta X_E + \Delta X_{ST} - \Delta X_P - \Delta X_S)^2 +$$
$$2(Y_E + Y_{ST} - Y_P - Y_S)(\Delta Y_E + \Delta Y_{ST} - \Delta Y_P - \Delta Y_S) + (\Delta Y_E + \Delta Y_{ST} - \Delta Y_P - \Delta Y_S)^2 +$$
$$2(Z_E + Z_{ST} - Z_P - Z_S)(\Delta Z_E + \Delta Z_{ST} - \Delta Z_P - \Delta Z_S) + (\Delta Z_E + \Delta Z_{ST} - \Delta Z_P - \Delta Z_S)^2\Big]$$

After making the following substitutions

$$\begin{aligned}
\underline{\overline{X}} &= (\Delta X_E + \Delta X_{ST} - \Delta X_P - \Delta X_S) \\
\underline{\overline{Y}} &= (\Delta Y_E + \Delta Y_{ST} - \Delta Y_P - \Delta Y_S) \\
\underline{\overline{Z}} &= (\Delta Z_E + \Delta Z_{ST} - \Delta Z_P - \Delta Z_S) \\
R_x &= (X_E + X_{ST} - X_P - X_S) \\
R_y &= (Y_E + Y_{ST} - Y_P - Y_S) \\
R_z &= (Z_E + Z_{ST} - Z_P - Z_S)
\end{aligned}$$

the range at time 2 becomes a function of $\underline{\overline{X}}$, $\underline{\overline{Y}}$, and $\underline{\overline{Z}}$:

$$r_2 = f(\underline{\overline{X}}, \underline{\overline{Y}}, \underline{\overline{Z}}) \qquad = \sqrt{[r_1^2 + 2R_x\underline{\overline{X}} + 2R_y\underline{\overline{Y}} + 2R_z\underline{\overline{Z}} + \underline{\overline{X^2}} + \underline{\overline{Y^2}} + \underline{\overline{Z^2}}]}$$

After subtracting the ranges at time one and two, the zero order term of this expression is eliminated and the range difference becomes a Taylor series of directional derivatives

$$r_2 - r_1 = f_\alpha(0,0,0) \times \left|\vec{D}\right| + \frac{1}{2} f_{\alpha\alpha}(0,0,0) \times \left|\vec{D}\right|^2 + \frac{1}{6} f_{\alpha\alpha\alpha}(0,0,0) \times \left|\vec{D}\right|^3$$

where the change vector is

$$\vec{D} = \underline{\overline{X}}\vec{i} + \underline{\overline{Y}}\vec{j} + \underline{\overline{Z}}\vec{k}$$

and the unit change vector is

$$\vec{\alpha} = \frac{\underline{\overline{X}}}{\left|\vec{D}\right|}\vec{i} + \frac{\underline{\overline{Y}}}{\left|\vec{D}\right|}\vec{j} + \frac{\underline{\overline{Z}}}{\left|\vec{D}\right|}\vec{k}$$

If,

$$a = \frac{\overline{X}}{\left|\vec{D}\right|}$$
$$b = \frac{\overline{Y}}{\left|\vec{D}\right|}$$
$$c = \frac{\overline{Z}}{\left|\vec{D}\right|}$$

then the derivatives are

$$f_\alpha = \frac{(aR_x + bR_y + cR_z)}{r_1}$$
$$f_{\alpha\alpha} = \frac{r_1 - \frac{1}{r_1}(aR_x + bR_y + cR_z)}{r_1^2}$$
$$f_{\alpha\alpha\alpha} = \frac{r_1(aR_x + bR_y + cR_z)(-1 - 2(a+b+c)) + \frac{3}{r_1}(aR_x + bR_y + cR_z)^3}{r_1^4}$$

To evaluate the series, the position differences between time one and time two are needed for the following quantities: earth-centered station coordinates (barycentric frame), planet-centered satellite coordinates (barycentric frame) and barycentric planet coordinates. The change in station and satellite coordinate values can be obtained to the required degree of precision through simple differencing. However, the barycentric planet positions require the use of a Chebyshev polynomial differencing scheme.

Planet positions are obtained from the JPL planetary ephemeris tapes which contain Chebyshev coefficients, The degree and period of validity for these coefficients vary depending on the body of interest. The Chebyshev polynomials, which are a function of normalized time over the period of validity are multiplied by their corresponding Chebyshev coefficients and summed overall degrees to compute the barycentric planet position. However, as discussed earlier, a simple differencing of the coordinates computed in this fashion leads to a loss of numerical significance. Another approach is to compute the range difference directly by first differencing like-degree Chebyshev polynomials at times 1 and 2, multiplying by the appropriate Chebyshev coefficient, and finally summing over all degrees. This method, however, also loses some, (but less) numerical significance, both in the higher order terms of the Chebyshev polynomials and in the differencing of the normalized times.

Therefore, the new version computes the change in planet positions by differencing times 1 and 2 before normalization and by using the following recursive Chebyshev polynomial formulation:
Δt = time tags differenced then normalized
$\Delta C_0 = 0$
$\Delta C_t = \Delta t$
$\Delta C_i = 2(t_2 \times \Delta C_{i-1} + \Delta t \times \Delta C_{i-1})$ - ΔC_{i-2}

The major problem with this new method is that it assumes that all time tags for each measurement (and all measurements in a block) fall within the same JPL time period. In practice, however, measurement blocks, or even a single measurement, can span two separate sets of Chebyshev coefficients. Therefore, tests were run to examine the behavior of Chebyshev polynomials and their coefficients near the period boundaries in hopes of calculating a new set of coefficients when there is the possibility of a period split. A least square method proved effective in solving for a new set of coefficients which are valid throughout the period boundary region and, thus, the entire measurement block. This method is accurate to approximately 1mm in range and 2e-8 m/s in range rate. To save time, the matrix multiplications and inversions for the right hand side of the least squares equation can be calculated once per run. In addition, the coefficients are rotated from the 12000 to the true of reference coordinate system to save computations at measurement times. In addition to the improved numerical precision, the calculation of planet positions and velocities has been vectorized and simplified. Previously, GEODYN computed positions for all the planets and the sun in a non-vector fashion. In the updated version GEODYN performs the vectorized position calculations only for those bodies needed in the model (Sun, Earth, Moon, Jupiter, Saturn, and the central body).

RESULTS

A series of tests have been run in hopes of testing precision on the basis of the repeatability of computations. These tests involve first simulating data at varying Doppler counting intervals and then using this simulated data as .input for a second run. Ideally, the second run's residuals would be zero. However, the finite level of computer and software precision induce numerical error, causing non-zero residuals. The purpose of this work has been to implement a new computational to reduce this numerical error to an acceptable level for the upcoming Mars Observer.

First, the old version of GEODYN simulated 842 two-way Doppler measurements at a mixture of both 60 and 600 second counting intervals. This data was processed in a second run of one iteration. This process was repeated at a 10 second counting interval since the Mars Observer data is anticipated to be at 10 second intervals. Next, the updated version of GEODYN was put through the same tests. The results are as follows:

Count Interval(sec)	Version	Max Residual (cm/s)	RMS (cm/s)
60 /600	Old	0.017121	0.0046
	New	0.000005	0.0000
10	Old	0.078125	0.0313
	New	0.000035	0.0000

Tests were also run with a perturbed initial state using the 60/600 second simulated data as input. The X, Y, and Z coordinates were all perturbed by 100 meters and GEODYN was allowed to run until convergence. The final state vector was compared to the a priori state used to simulate the data. Ideally, the two states would be the same but, again, numerical error causes a discrepancy.

Version	DX	DY	DZ	DXDOT	DYDOT	DZDOT
Old	0.872	0.283	1.779	0.0000356	0.0000372	0.0000524
New	2e-4	1e-4	5.0e-3	0.0000000	0.0000000	0.0000000

Obviously, our efforts have been successful. The new method gives precision well within the Mars Observer mission requirements. Also, it does this at no cost in run time. The new version has added many computations but, because of efforts in vectorization and efficient calculation, the old and new CPU times are comparable. These test cases have been run with many measurement blocks consisting of just one member. When multi-member measurement blocks arc run, we expect to see even further improvement in run time.

Altimeter height

The observed altimeter height is actually a measure of the height of the satellite above the sea surface. Mathematically this may be represented as

$$H_{alt} = H_{ellipsoidal} - H_{geoid} - H_{earth\ \&\ oceantides} - H_{bias} \qquad (6.1 - 6)$$

$H_{ellipsoidal}$ is obtained through successive applications of the following iterative procedures.

From Figure 5.1- I, which shows a cross-section of the earth's ellipsoid, we can define the flattening of the ellipse cross-section as:

$$\frac{f = a - b}{a} \qquad (6.1 - 7)$$

where a and b are the two axes of the ellipse. The eccentricity e of the ellipse is then

$$e^2 = 1 - (1 - f)^2 \qquad (6.1 - 8)$$

For a point with x, y, z earth fixed Cartesian coordinates we can find its height It above the ellipsoid in the following manner. In general

$$h << N \qquad (6.1 - 9)$$

and since the earth is nearly a sphere

$$e << 1 \tag{6.1 - 10}$$

therefore

$$N \sin\phi \simeq Z \tag{6.1 - 11}$$

again from Figure 5.1-1. We also have from this figure that

$$t = Ne^2 \sin\phi \tag{6.1 - 12}$$

and therefore

$$t \simeq e^2 Z \tag{6.1 - 13}$$

This is the initial approximation for t.
The iterative procedure then uses

$$Z_t = Z + t \tag{6.1 - 14}$$

$$N + h = (x^2 + y^2 + z_t^2)^{\frac{1}{2}} \tag{6.1 - 15}$$

$$\sin\phi = \frac{Z_t}{(N+h)} \tag{6.1 - 16}$$

$$N = \frac{a}{(1 - e^2 \sin^2\phi)^{\frac{1}{2}}} \tag{6.1 - 17}$$

and finally

$$t = Ne^2 \sin\phi \tag{6.1 - 18}$$

This new value for t is then used as the new starting point, and the process is repeated. When t converges to within 10^{-9} meters, equations (6.1-15), (6.1-16) and (6.1 17) are used to find h which gives $H_{ellipsoldal}$. The rest of the terms in equation (6.1-6) are:

H_{geoid} - the geoid height due to spherical harmonic coefficients (Section 5.7.1) and gravity anomalies (Section 5.7.2).
$H_{earth\ \&\ ocean\ tide}$ - the earth tidal bulge due to solid earth tides (Section 5.5) and ocean tides
H_{bias} - the altitude bias caused by uncertainties in the alignment of the spacecraft altitude and uncertainties in the local nadir (Section 7.8).

Right ascension and declination
The topocentric right ascension α and declination δ are inertial coordinate system measurements as illustrated in Figure 6.1-1. GEODYN computes these angles from the components

of the Earth-fixed station-satellite vector and the Greenwich hour angle θ_g.

$$\alpha = \tan^{-1}\left[\frac{\rho_2}{\rho_1}\right] + \theta_g \tag{6.1 - 19}$$

$$\delta = \sin^{-1}\left[\frac{\rho_3}{\rho}\right] \tag{6.1 - 20}$$

The remaining measurements are in the topocentric horizon coordinate system. These all require the $\hat{N}$, $\hat{Z}$, and $\hat{E}$ (north, zenith, and east base line) unit vectors which describe the coordinate system.

Direction cosines

There arc three direction cosines associated with the station-satellite vector in the topocentric system. These are:

$$\begin{aligned} l &= \hat{u} \cdot \hat{E} \\ m &= \hat{u} \cdot \hat{N} \\ n &= \hat{u} \cdot \hat{Z} \end{aligned} \tag{6.1 - 21}$$

The l and m direction cosines are observation types for GEODYN.

X and Y angles

There are two types of X - Y angles, depending on the type of antenna mount of the receivers. The X and Y angles for a north - south mount are illustrated in Figure 6.1-2. They are computed by

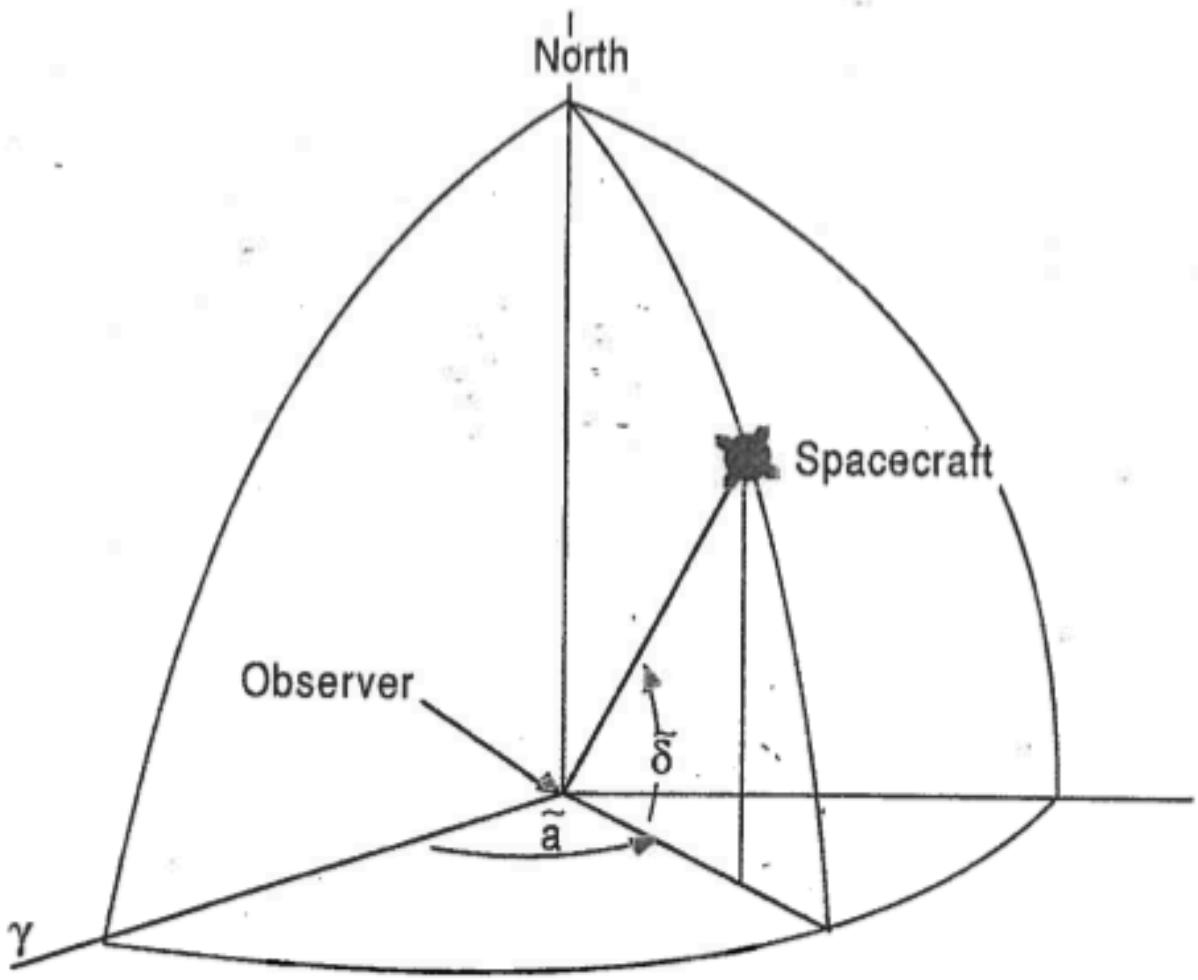
$$X_a = \tan^{-1}\left[\frac{l}{n}\right] \tag{6.1 - 22}$$

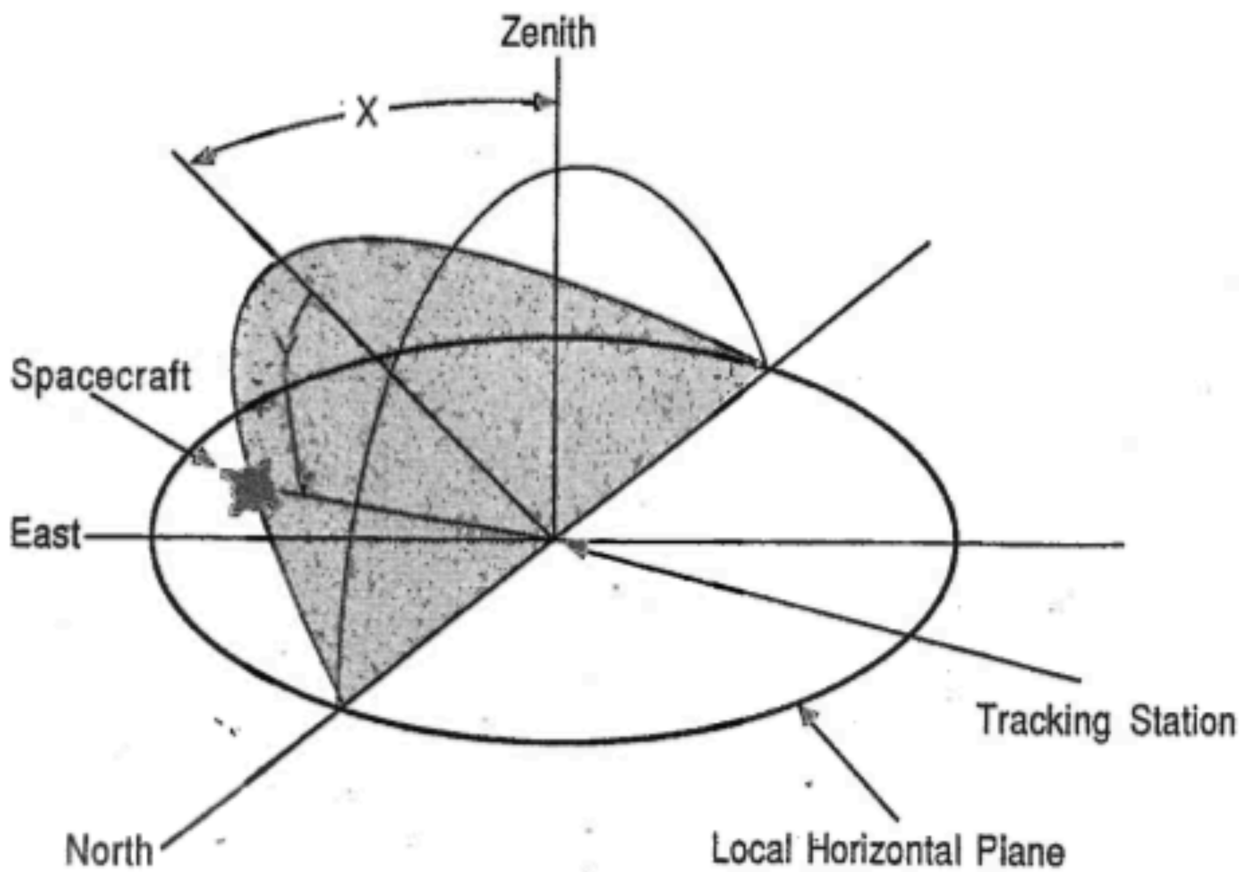
$$Y_a = \sin^{-1}(m) \tag{6.1 - 23}$$

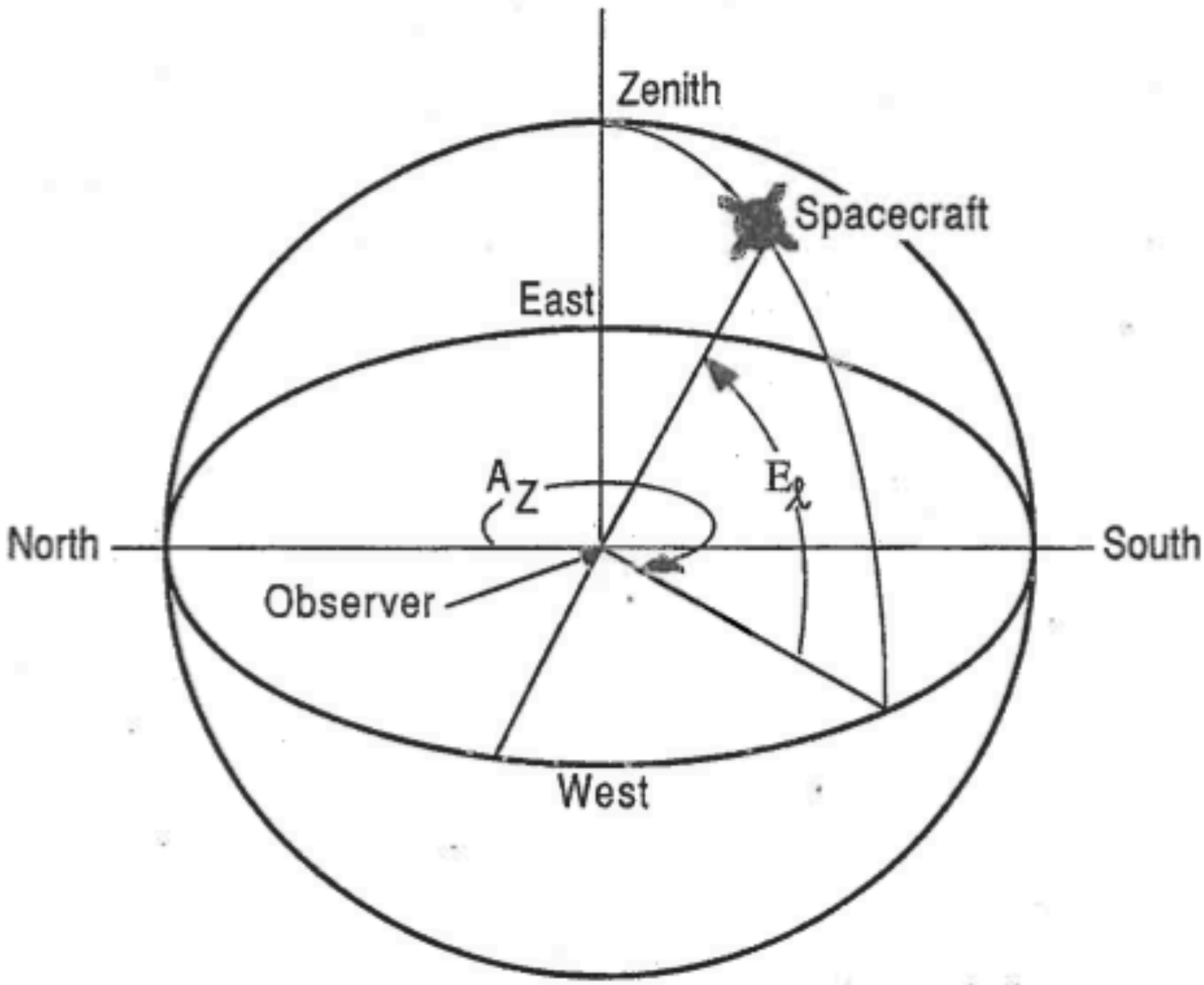
Figure 6.1-3 illustrates the measurements of azimuth and elevation. These angles are computed by:

$$A_z = \tan^{-1}\left[\frac{l}{m}\right] \tag{6.1 - 24}$$

$$E_l = \sin^{-1}(n) \tag{6.1 - 25}$$


North
Spacecraft
Observer
$\tilde{\delta}$
$\tilde{a}$
γ


Zenith
X
Y
Spacecraft
East
Tracking Station
North
Local Horizontal Plane


Zenith
Spacecraft
East
A
Z
E
North
South
Observer
West

6.2 THE GEOMETRIC PARTIAL DERIVATIVES

The partial derivatives for each of the calculated geometric measurements with respect to the satellite positions and velocity are given here. All are in the geocentric, Earth-fixed system. The r_i refer to the Earth-fixed components of $\bar{r}$.

Range

$$\frac{\partial \rho}{\partial r_i} = \frac{\rho_i}{\rho} \quad where\ \rho_i = X_i - X_i\ station \qquad (6.2 - 1)$$

The partial derivative of the range with respect to the speed of light is c given by:

$$\frac{\partial \rho}{\partial c} = \frac{\rho}{c} \qquad (6.2 - 1a)$$

Range rate

$$\frac{\partial \dot{\rho}}{r_i} = \frac{1}{\rho}\left[\dot{r}_i - \frac{\dot{\rho}\rho_i}{\rho}\right] \qquad (6.2 - 2)$$

$$\frac{\partial \dot{\rho}}{\partial \dot{r}_i} = \frac{\rho_i}{\rho} \qquad (6.2 - 3)$$

The partial derivative of the range rate with respect to the speed of light is given by:

$$\frac{\partial \dot{\rho}}{\partial c} = \frac{\dot{\rho}}{c} \qquad (6.2 - 3a)$$

Altimeter

For the purpose of computing these partial derivatives the following formula developed in Section 5.1 is used

$$H_{alt} = r - a_e - \frac{3}{2}a_e f^2 (\frac{z}{r})^4 + (a_e f + \frac{3}{2}a_e f^2)(\frac{z}{r})^2 \qquad (6.2 - 4)$$

where
a_e = the earth's mean equatorial radius
f = the earth's flattening
$z = r_3$, the z component of the earth's fixed satellite vector.

The partial derivatives for altimeter range are then given by:

$$\frac{\partial H_{alt}}{\partial r_i} = \frac{r_i}{r} + \frac{1}{r}\left[(2a_e f + 3a_e f^2)(\frac{z}{r}) - 6a_e f^2 (\frac{z}{r})^3\right]\left[\frac{\partial z}{\partial r_i} - \frac{z r_i}{r^2}\right] \qquad (6.2 - 5)$$

The partial derivative of the altimeter range with respect to the speed of light is

$$\frac{\frac{\partial H_{alt}}{\partial c} = H_{alt}}{c} \tag{6.2 - 5a}$$

The partial derivative of the altimeter range with respect to a_e is

$$\frac{\partial H_{alt}}{\partial a_e} = -\sqrt{1 - \varepsilon \sin^2 \phi} \tag{6.2 - 5b}$$

where:
ε = 1 - $(1 - f)^2$
and
ϕ = satellite latitude

Since the altimeter measurement depends on the spherical harmonic coefficients and on the gravity ?anomalies explicitly, the total partial with respect to any parameter for this measurement has the form

$$\frac{\partial H_{alt}}{\partial \beta} = \frac{\partial H_{alt}}{\partial \bar{r}} \frac{\partial \bar{r}}{\partial \beta} + \frac{\partial H_{alt}}{\partial H_{geoid}} \frac{\partial H_{geoid}}{\partial \beta} \tag{6.2 - 6}$$

where

$\frac{\partial \bar{r}}{\partial \beta}$ are the variational equations obtained through numerical integration.

and

$\frac{\partial H_{alt}}{\partial H_{geoid}} \simeq$ -1 (neglecting tilt in the local vertical)

and

$\frac{\partial H_{geoid}}{\partial \beta}$ where β maybe geopotential coefficients or gravity anomalies

For geopotential coefficients

$$\frac{\partial H_{geoid}}{\partial C_n^m} = a_e \cos(m\lambda) P_n^m(\sin \phi)$$
$$\frac{\partial H_{geoid}}{\partial S_n^m} = a_e \sin(m\lambda) P_n^m(\sin \phi)$$

For gravity anomalies

$$\frac{\partial H_{geoid}}{\partial \Delta g} = \frac{R}{4\pi GM} \int \int_{\sigma} S(\Psi) d\delta$$

where $S(\Psi)$ is the Stokes' function given by

$$S(\Psi) = \frac{1}{\sin(\frac{\Psi}{2})} - 6\sin(\frac{\Psi}{2}) + 1 - 5\cos\Psi - 3\cos\Psi log_e(\sin(\frac{\Psi}{2}) + \sin^2(\frac{\Psi}{2}))$$

Right Ascension

$$\frac{\partial \alpha}{\partial r_1} = \frac{-\rho_2}{\rho_1^2 + \rho_2^2} \quad (6.2 - 7)$$

$$\frac{\partial \alpha}{\partial r_2} = \frac{\rho_1}{\rho_1^2 + \rho_2^2} \quad (6.2 - 8)$$

$$\frac{\partial \alpha}{\partial r_3} = 0 \quad (6.2 - 9)$$

Declination

$$\frac{\partial \delta}{\partial r_1} = \frac{-\rho_1 \rho_3}{\rho^2 \sqrt{\rho_1^2 + \rho_2^2}} \quad (6.2 - 10)$$

$$\frac{\partial \delta}{\partial r_2} = \frac{-\rho_2 \rho_3}{\rho \sqrt{\rho_1^2 + \rho_2^2}} \quad (6.2 - 11)$$

$$\frac{\partial \delta}{\partial r_3} = \frac{\sqrt{\rho_1^2 + \rho_2^2}}{\rho^2} \quad (6.2 - 12)$$

Direction Cosines

$$\frac{\partial l}{\partial r_i} = \frac{1}{\rho}[E_i - lu_i] \quad (6.2 - 13)$$

$$\frac{\partial m}{\partial r_i} = \frac{1}{\rho}[N_i - mu_i] \quad (6.2 - 14)$$

$$\frac{\partial n}{\partial r_i} = \frac{1}{\rho}[Z_i - nu_i] \quad (6.2 - 15)$$

X and Y Angles (For North - South mount)

$$\frac{\partial X_a}{\partial r_i} = \frac{nE_i - lZ_i}{\rho(1 - m^2)} \quad (6.2 - 16)$$

$$\frac{\partial Y_a}{\partial r_i} = \frac{-(N_i - mu_i)}{\rho\sqrt{1 - m^2}} \quad (6.2 - 17)$$

Azimuth and Elevation

$$\frac{\partial A_z}{\partial r_i} = \frac{mE_i - lN_i}{\rho\sqrt{1-n^2}} \tag{6.2 - 18}$$

$$\frac{\partial E_l}{\partial r_i} = \frac{(Z_i - nu_i)}{\rho(1-n^2)} \tag{6.2 - 19}$$

6.3 THE TIME DERIVATIVES

The derivatives of each measurement type with respect to time is presented below. All are in the Earth-fixed system
Range

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{r}} \tag{6.3 - 1}$$

Range Rate
The range rate derivative deserves special attention. Remembering that

$$\dot{\bar{\rho}} = \dot{\bar{r}}, \tag{6.3 - 2}$$

We write

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{\rho}} \tag{6.3 - 3}$$

Thus

$$\ddot{\rho} = \dot{\hat{u}} \cdot \dot{\bar{\rho}} + \hat{u} \cdot \ddot{\bar{\rho}} \tag{6.3 - 4}$$

Because

$$\dot{\bar{\rho}} = \frac{d}{dt}(\rho\hat{u}) = \rho\dot{\hat{u}} + \dot{\rho}\hat{u} \tag{6.3 - 5}$$

we may substitute in Equation 4 above for $\hat{u}$:

$$\ddot{\rho} = \frac{1}{\rho}(\dot{\bar{\rho}} \cdot \dot{\bar{\rho}} - \rho\dot{\hat{u}} \cdot \dot{\bar{\rho}}) + \hat{u} \cdot \ddot{\bar{\rho}} \tag{6.3 - 6}$$

or, as

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{\rho}} \qquad (6.3 - 7)$$

we may write:

$$\ddot{\rho} = \frac{1}{\rho}(\dot{\bar{\rho}} \cdot \dot{\bar{\rho}} - \dot{\rho}^2 + \bar{\rho} \cdot \ddot{\bar{\rho}}) + \hat{u} \cdot \ddot{\bar{\rho}} \qquad (6.3 - 8)$$

In order to obtain $\ddot{\bar{\rho}}$, we use the limited gravity potential (see Section 8.3).

$$U = \frac{GM}{r}\left[1 - \frac{C_{20}a_e^2}{r^2}P_2^0(\sin\phi)\right] \qquad (6.3 - 9)$$

The gradient of this potential with respect to the Earth-fixed position coordinates of the satellite is the part of $\ddot{\bar{\rho}}$ due to the geopotential:

$$\frac{\partial U}{\partial r_i} = -\frac{GM}{r^3}\left[1 - \frac{3a_e^2C_{20}}{2r^2}[5\sin^2\phi - 1 - 2\frac{z}{r_i}]\right]r_i \qquad (6.3 - 10)$$

We must add to this the effect of the rotation of the coordinate system. (The Earth-fixed coordinate system rotates with respect to the true of date coordinates with a rate θ_g, the time rate of change of the Greenwich hour angle.)
The components of $\ddot{\bar{\rho}}$ are then

$$\ddot{\rho}_1 = \frac{\partial U}{\partial r_1} + [\dot{x}\cos\theta_g + \dot{y}\sin\theta_g]\dot{\theta}_g + \dot{r}_2\dot{\theta}_g \qquad (6.3 - 11)$$

$$\ddot{\rho}_2 = \frac{\partial U}{\partial r_2} + [-\dot{x}\sin\theta_g + \dot{y}\cos\theta_g]\dot{\theta}_g - \dot{r}_1\dot{\theta}_g \qquad (6.3 - 12)$$

$$\ddot{\rho}_3 = \frac{\partial U}{\partial r_3} = \frac{\partial U}{\partial z} \qquad (6.3 - 13)$$

The bracketed quantities, above correspond to the coordinate transformations coded in subroutines XEFIX and YEFIX. These transforms are used on the true of date satellite velocity components $\dot{x}$ and $\dot{y}$. The interested reader should refer to Section 3.4 for further information on transformations between Earth-fixed and true of date coordinates.

It should be noted that-all quantities in this formula, with the exception of those quantities bracketed, are Earth-fixed values. (The magnitude r is invariant with respect to the

coordinate system transformations.) The remaining time derivatives are tabulated here:
Right ascension:

$$\dot{\alpha} = \frac{u_1\dot{r}_2 - u_2\dot{r}_1}{\rho(1 - u_3^2)} \tag{6.3 - 14}$$

Declination:

$$\dot{\delta} = \frac{\dot{r}_3 - \dot{\rho}u_3}{\rho\sqrt{1 - u_3^2}} \tag{6.3 - 15}$$

Direction Cosines:

$$\dot{l} = \frac{\bar{\rho} \cdot \hat{E} - l\dot{\rho}}{\rho} \tag{6.3 - 16}$$

$$\dot{m} = \frac{\dot{\bar{\rho}} \cdot \hat{N} - m\dot{\rho}}{\rho} \tag{6.3 - 17}$$

X and Y angles:

$$\dot{X}_a = \frac{\dot{\bar{\rho}} \cdot (n\hat{E} - l\hat{Z})}{\rho(1 - m^2)} \tag{6.3 - 18}$$

$$\dot{Y}_a = \frac{\dot{\bar{\rho}} \cdot \hat{N} - m\dot{\rho}}{\rho\sqrt{1 - m^2}} \tag{6.3 - 19}$$

Azimuth:

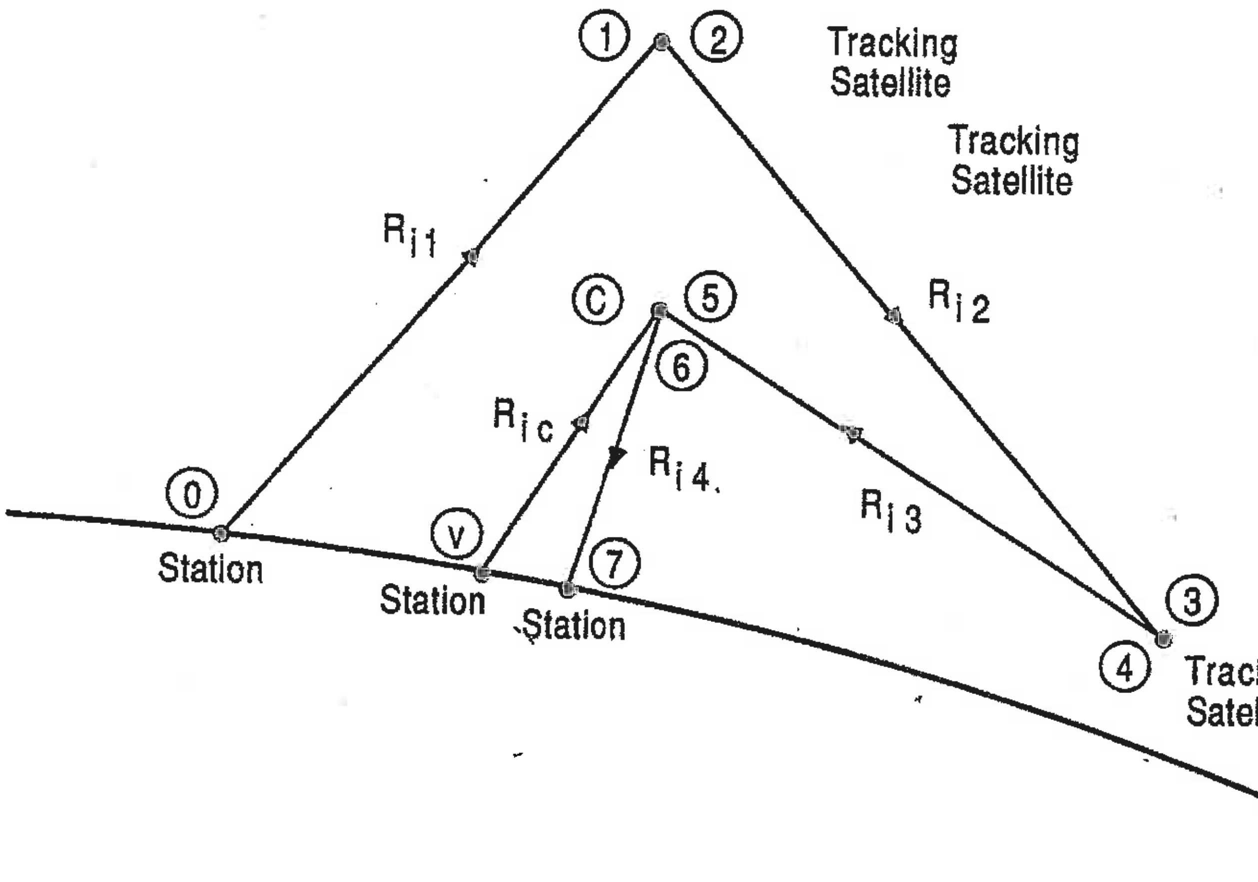
$$\dot{A}_z = \frac{\dot{\bar{\rho}} \cdot (m\hat{E} - l\hat{N})}{\rho(1 - m^2)} \tag{6.3 - 20}$$

(6.3 - 21)

Elevation:

$$\dot{E}_l = \frac{\dot{\bar{\rho}} \cdot \hat{Z} - m\dot{\rho}}{\rho\sqrt{1 - m^2}} \tag{6.3 - 22}$$

Figure 10. Figure 6.4-1 Examining the SST Signal Paths in Inertial Space: SST --- Satellite-to-Satellite



6.4 MATHEMATICAL MODEL FOR SST MEASUREMENTS

In this section the algorithms for the satellite-to-satellite measurements are developed. The SST signa! path in inertial space is defined, and the individual legs of the signal path are investigated. The SST range, range rate and Doppler count interval measurements are determined.

6.4.1 Definition of SST Terms

In order to model the SST measurements it is necessary to analyze the SST signal path. Figure 6.4-1 shows the SST signal path in inertial space. The uplink and downlink signal paths are illustrated and the individual legs 0: the signal path are labeled. The definitions of the satellite-to-satellite range links and associated times are presented on the following pages.
where
$\bar{\rho}_{TKNG}(t)$ is the inertial position of the tracking satellite at the time is the inertial position of the tracked satellite at time (t).
$\bar{\rho}_{TKED}(t)$ is the inertial station location at the time (t).
C is the speed of light.

then the station-satellite and the satellite-satellite range links are defined as follows:

$$R_{01} = |\bar{\rho}_{TKNG}(t_{01}) - \bar{\rho}_{sta}(t_{00})| \tag{6.4 - 1}$$
$$R_{02} = |\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})|$$
$$R_{03} = |\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})|$$
$$R_{04} = |\bar{\rho}_{sta}(t_{07}) - \bar{\rho}_{TKNG}(t_{06})|$$
$$R_{0c} = |\bar{\rho}_{TKNG}(t_{0c}) - \bar{\rho}_{sta}(t_{0v})|$$
$$R_{11} = |\bar{\rho}_{TKNG}(t_{11}) - \bar{\rho}_{sta}(t_{10})| \tag{6.4 - 2}$$
$$R_{12} = |\bar{\rho}_{TKED}(t_{13}) - \bar{\rho}_{TKNG}(t_{12})|$$
$$R_{13} = |\bar{\rho}_{TKNG}(t_{15}) - \bar{\rho}_{TKED}(t_{14})|$$
$$R_{14} = |\bar{\rho}_{sta}(t_{17}) - \bar{\rho}_{TKNG}(t_{16})|$$
$$R_{1c} = |\bar{\rho}_{TKNG}(t_{1c}) - \bar{\rho}_{sta}(t_{1v})|$$

Where the times are defined as follows:

t_{07} is the time at which counter is started

t_{17} is the time at which counter is stopped

$t_{16} = t_{17} - \frac{R_{14}}{C}$

$t_{06} = t_{07} - \frac{R_{04}}{C}$

$\tau_{15} = t_{16} - t_{15}$ = transponder delay time from receipt of signal at 5 and t_{15} until retransmission at 6.

$\tau_{05} = t_{06} - t_{05}$ = transponder delay time from receipt of signal at 5 and t_{05} until retransmission at 6.

$t_{14} = t_{15} - \frac{R_{13}}{C}$

$t_{02} = t_{03} - \frac{R_{02}}{c}$

$\tau_{11} = t_{12} - t_{11}$ = transponder delay 1 to 2 at = transponder delay 1 to 2 at t_{11}

$\tau_{01} = t_{02} - t_{01}$ = transponder delay 1 to 2 at = transponder delay 1 to 2 at t_{01}

$t_{10} = t_{11} - \frac{R_{11}}{C}$

$t_{00} = t_{01} - \frac{R_{01}}{C}$

$\tau_{1c} = t_{16} - t_{1c}$ = transponder delay C to 6 at t_{1c}

$\tau_{0c} = t_{06} - t_{06}$ = transponder delay C to 6 at t_{0c}

$t_{1v} = t_{1c} - \frac{R_{1c}}{C}$

$t_{0v} = t_{0v} - \frac{R_{0c}}{C}$

6.4.2 SST Range Measurements and Partials

Range measurement

$$R = \frac{1}{2}[|\bar{\rho}_{TKNG}(t_{01}) - \bar{\rho}_{STA}(t_{00})| + |\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})| + |\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})| + |\bar{\rho}_{STA}(t_{07}) - \bar{\rho}_{TKNG}(t_{06})|] \quad (6.4 - 3)$$

$$R = \frac{1}{2}[R_{01} + R_{02} + R_{03} + R_{04}] \quad (6.4 - 4)$$

Range partials

$$\frac{\partial R}{\partial \rho_{TKNG}} = \frac{1}{2}\frac{\partial}{\partial \rho_{TKNG}}[|\bar{\rho}_{TKNG}(t_{01}) - \bar{\rho}_{STA}(t_{00})| + |\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})| + |\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})| + |\bar{\rho}_{STA}(t_{04})| + |\bar{\rho}_{STA}(t_{07}) - \bar{\rho}_{TKNG}(t_{06})|]$$

(6.4 - 5)

$$\frac{\partial R}{\partial \bar{\rho}_{TKNG_0}} = \frac{1}{2}\left[\frac{\partial|\bar{\rho}_{TKNG}(t_{01}) - \bar{\rho}_{STA}(t_{00})|}{\partial \rho_{TKNG}(t_{01})}\frac{\partial \bar{\rho}_{TKNG}(t_{01})}{\partial \bar{\rho}_{TKNG_0}} + \frac{\partial|\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})|}{\partial \bar{\rho}_{TKNG}(t_{02})}\frac{\partial \bar{\rho}_{TKNG}(t_{02})}{\partial \bar{\rho}_{TKNG_0}} + \frac{\partial|\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})|}{\partial \bar{\rho}_{TKNG}(t_{05})}\frac{\partial \bar{\rho}_{TKNG}(t_{05})}{\partial \bar{\rho}_{TKNG_0}} + \frac{\partial|\bar{\rho}_{STA}(t_{07}) - \bar{\rho}_{TKNG}(t_{06})|}{\partial \rho_(\bar{t}_06)}\frac{\partial \bar{\rho}_{TKNG}(t_{06})}{\partial \bar{\rho}_{TKNG}}\right] \quad (6.4 - 6)$$

$$\frac{\partial R}{\partial \bar{\rho}_{TKED_0}} = \frac{1}{2}\frac{\partial}{\partial \rho_{TKED_0}}[|\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})| + |\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})|] \quad (6.4 - 7)$$

$$\frac{\partial R}{\partial \bar{\rho}_{TKED_0}} = \frac{1}{2}\left[\frac{\partial|\bar{\rho}_{TKED}(t_{03}) - \bar{\rho}_{TKNG}(t_{02})|}{\partial \rho_{TKED}(t_{03})}\frac{\partial \bar{\rho}_{TKED}(t_{03})}{\partial \bar{\rho}_{TKED_0}} + \frac{\partial|\bar{\rho}_{TKNG}(t_{05}) - \bar{\rho}_{TKED}(t_{04})|}{\partial \bar{\rho}_{TKED}(t_{04})}\frac{\partial \bar{\rho}_{TKED}(t_{04})}{\partial \bar{\rho}_{TKNG_0}}\right] \quad (6.4 - 8)$$

6.4.3 SST Range Rate Measurements

The algorithms for the SST range rate and Doppler count interval measurements are developed in this section. The frequencies associated with the tracking station are defined, and the signal path is traced backwards from receiver to transmitter.

Defining the frequencies associated with the tracking station:

f_b = Ground generated carrier frequency (used to phase lock the tracking satellite VCO to ground)

f_{RR} = Ground generated ranging carrier frequency

f_{LO} = Ground generated receiver local oscillator frequency (this is the frequency which is beat against the incoming signal from the tracking satellite)

f_B = Ground generated bias frequency

J and K = Tracking satellite transponder frequency multipliers

$\frac{n}{m}$ = Tracked satellite transponder ratio

N_8 = Integrated Doppler plus bias cycles accumulated

T = Integration time interval required to accumulate N_8

Following backwards along the signal path (i.e., tracing the signal phase back along the signal path); the time of transmission of the of the last cycle counted is:

Along the 4-Way Path

$$t_{10} = t_{17} - \frac{R_{14}}{C} - \tau_{15} - \frac{R_{13}}{C} - \tau_{13} - \frac{R_{12}}{C} - \tau_{11} - \frac{R_{11}}{C} \quad (6.4 - 9)$$

Along the 2-Way Path

$$t_{1v} = t_{17} - \frac{R_{14}}{C} - \tau_{1c} - \frac{R_{1c}}{C} \quad (6.4 - 10)$$

Following backwards along the signal path (i.e., tracing the signal phase back along the signal path) the time of transmission of the beginning of the first cycle counted is

Along the 4-Way Path

$$t_{00} = t_{07} - \frac{R_{04}}{C} - \tau_{05} - \frac{R_{03}}{C} - \tau_{03} - \frac{R_{02}}{C} - \tau_{01} - \frac{R_{01}}{C} \quad (6.4 - 11)$$

Along the 2- Way Path

$$t_{0v} = t_{07} - \frac{R_{04}}{C} - \tau_{0c} - \frac{R_{0c}}{C} \quad (6.4 - 12)$$

The time interval of transmission for the 4-way link (T_0) is then

$$T_0 = t_{10} - t_{00} \quad (6.4 - 13)$$

and the time interval of transmission for the 2-way link (T_v) is

$$T_v = t_{1v} - t_{0v} \quad (6.4 - 14)$$

and the number of cycles transmitted in those periods is

For the Ranging Carrier,

$$N_{0RR} = T_0 f_{RR} \tag{6.4 - 15}$$

For the Locking Carrier 4-way term

$$N_{ob} = T_0 f_b \tag{6.4 - 16}$$

and
For the Locking Carrier 2way term

$$N_{vb} = T_v f_b \tag{6.4 - 17}$$

At the rocking satellite a multiple of the locking carrier is subtracted from the ranging carrier before retransmission at 2

$$N_2 = N_{ORR} - \frac{K - \frac{77}{4}}{K + 1.5} N_{ob} \tag{6.4 - 18}$$

The tracked satellite multiples the arriving carrier at 3 by $\frac{n}{m}$and retransmits at 4

$$N_4 = N_2 \frac{n}{m} \tag{6.4 - 19}$$

The signal arriving back at 5 at the tracked satellite has a multiple of the locking carrier C added to it before retransmission at 6.

$$N_6 = N_4 + \frac{J - 21}{K + 1.5} N_{vb} \tag{6.4 - 20}$$

The signal received at the station, 7 is beat against a local oscillator for the duration of the measurement T

$$N_8 = N_6 - T f_{LO} \tag{6.4 - 21}$$

N_8 represents the integral of the Doppler phase delay plus bias.

$$N_8 = \int_{t}^{t+T} f_d + f_B dt \tag{6.4 - 22}$$

$$N_8 = T f_B + \int_{t}^{t+T} f_d dt \tag{6.4 - 23}$$

The integrated Doppler phase delay N_9 is then

$$N_9 = \int_{t}^{t+T} f_d dt = N_8 - T f_B \tag{6.4 - 24}$$

Collecting terms then

$$N_9 = -T f_B - T f_{LO} + \frac{J - 21}{K + 1.5} N_{vb} + \frac{n}{m}\left[N_{ORR} - \frac{K - \frac{77}{4}}{K + 1.5} N_{ob}\right] \tag{6.4 - 25}$$

But since

$$\begin{aligned} N_{ORR} &= T_0 f_{RR} \\ N_{ob} &= T_0 f_b \\ N_{vb} &= T_v f_b \end{aligned} \tag{6.4 - 26}$$

Then

$$N_9 = -T(f_B + f_{LO}) + \frac{J - 21}{K + 1.5} f_b T_v + \frac{n}{m}\left[f_{RR} - \frac{K - \frac{77}{4}}{K + 1.5} f_b\right] T_0 \tag{6.4 - 27}$$

Defining

$$f_x = \frac{n}{m}\left[f_{RR} - \frac{K - \frac{77}{4}}{K + 1.5} f_b\right] \tag{6.4 - 28}$$

$$N_N = T(f_B + f_{LO}) \tag{6.4 - 29}$$

$$f_Y = \frac{J - 21}{K + 1.5} f_b \tag{6.4 - 30}$$

Then

$$N_9 = -N_N + f_Y T_v + f_X T_0 \tag{6.4 - 31}$$

Dividing through by f_x

$$\frac{N_9}{f_X} = -\frac{N_N}{f_X} + \frac{f_Y}{f_X} T_v + T_0 \tag{6.4 - 32}$$

Going back to the definitions of T_0 and T_v

$$\begin{aligned} T_0 = t_{10} - t_{00} = t_{17} - \frac{R_{14}}{C} - \tau_{15} - \frac{R_{13}}{C} - \tau_{13} - \frac{R_{12}}{C} - \tau_{11} - \frac{R_{11}}{C} \\ -\tau_{07} + \frac{R_{04}}{C} + \tau_{05} + \frac{R_{03}}{C} + \tau_{03} + \frac{R_{02}}{C} + \tau_{01} + \frac{R_{01}}{C} \end{aligned} \tag{6.4 - 33}$$

but

$$t_{17} - t_{07} = T$$

therefore

$$\begin{aligned} T_0 = T - \frac{(R_{14} + R_{13} + R_{12} + R_{11}) - (R_{04} + R_{03} + R_{02} + R_{01})}{C} \\ + \tau_{05} - \tau_{15} + \tau_{03} - \tau_{13} + \tau_{01} - \tau_{11} \end{aligned} \tag{6.4 - 34}$$

And

$$\begin{aligned} T_v &= t_{1v} - t_{0v} \\ &= t_{17} - t_{07} - \frac{R_{14}}{C} - \tau_{1c} - \frac{R_{1c}}{C} + \frac{R_{04}}{C} + \tau_{0c} + \frac{R_{0c}}{C} \end{aligned}$$

$$T_v = T - \frac{(R_{14} + R_{1c}) - (R_{04} + R_{0c})}{C} + \tau_{0c} - \tau_{1c} \tag{6.4 - 35}$$

Let

$$D_4 = \tau_{05} - \tau_{15} + \tau_{03} - \tau_{13} + \tau_{01} - \tau_{11} \tag{6.4 - 36}$$

$$D_2 = \tau_{0c} - \tau_{1c} \tag{6.4 - 37}$$

The Doppler equation

$$\frac{N_9}{f_X} = -\frac{N_N}{f_X} + \frac{f_Y}{f_X} T_v + T_0 \qquad (6.4 - 38)$$

becomes

$$\begin{aligned}\frac{N_9}{f_X} = -\frac{N_N}{f_X} + \frac{f_Y}{f_X}\left[T - \frac{(R_{14}+R_{1c})-(R_{04}+R_{0c})}{C} + D_2\right] + \\ T - \frac{(R_{14}+R_{13}+R_{12}+R_{11})-(R_{04}+R_{03}+R_{02}+R_{01})}{C} + D_4\end{aligned} \qquad (6.4 - 39)$$

Multiplying through by $\frac{C}{T}$ and rearranging terms

$$\begin{aligned}C\frac{(N_9+N_N)}{f_X T} = C\left[1+\frac{f_Y}{f_X}\right] - \frac{f_Y}{f_X}\frac{(R_{14}+R_{1c})-(R_{04}+R_{0c})}{T} + \frac{D_2 C}{T}\frac{f_Y}{f_X} \\ -\frac{(R_{14}+R_{13}+R_{12}+R_{11})-(R_{04}+R_{03}+R_{02}+R_{01})}{T} + \frac{D_4 C}{T}\end{aligned} \qquad (6.4 - 40)$$

$$\begin{aligned}\frac{N_9 C}{f_X T} + \frac{N_N C}{f_X T} - C - \frac{f_Y}{f_X}C = \frac{C}{T}\left[-\frac{f_Y}{f_X}D_2 + D_4\right] \\ -\frac{(R_{14}+R_{13}+R_{12}+R_{11})-(R_{04}+R_{03}+R_{02}+R_{01})}{T} \\ -\frac{f_Y}{f_X}\frac{(R_{14}+R_{1c})-(R_{04}+R_{0c})}{T}\end{aligned} \qquad (6.4 - 41)$$

Examining the left-hand side minus Doppler (i.e., less $\frac{N_{9C}}{f_x T}$)

$$\begin{aligned}-C\frac{f_Y}{f_X} + \frac{N_N C}{f_X T} - C = \frac{-Cf_Y + \frac{N_N C}{T}C - f_X C}{f_X} \\ = -\frac{C}{f_X}\left[\frac{J-21}{K+1.5}f_b - f_B - f_{LO} + \frac{n}{m}\left[f_{RR} - \frac{K-\frac{77}{4}}{K+1.5}f_b\right]\right]\end{aligned} \qquad (6.4 - 42)$$

However the Doppler bias f_B is chosen such that for the zero Doppler case

$$f_B = \frac{n}{m}\left[\frac{\frac{77}{4}-K}{K+1.5}f_b + f_{RR}\right] + \frac{(J-21)f_b}{K+1.5} - f_{LO} \qquad (6.4 - 43)$$

Substituting this back in, it can be readily seen that the term

$$-C\frac{f_Y}{f_X}+\frac{N_N C}{f_X T}-C=0 \tag{6.4 - 44}$$

Therefore

$$\frac{N_9 C}{f_X T}=\frac{C}{T}\left[\frac{f_Y}{f_X}D_2+D_4\right]-\frac{(R_{14}+R_{13}+R_{12}+R_{11})-(R_{04}+R_{03}+R_{02}+R_{01})}{T} \\ -\frac{f_Y}{f_X}\frac{(R_{14}+R_{1c})-(R_{04}+R_{0c})}{T} \tag{6.4 - 45}$$

Defining range rate on the long and short signal path as

$$\dot{R}_S=\dot{R}_C+\dot{R}_4 \\ \dot{R}_L=\dot{R}_1+\dot{R}_2+\dot{R}_3+\dot{R}_4 \tag{6.4 - 46}$$

Then the integral of the range rate along the signal paths

$$\int_{t_{07}}^{t_{17}}K\dot{R}_S dt+\int_{t_{07}}^{t_{17}}\dot{R}_L dt=K\left[\int_{t_{0C}}^{t_{1C}}\dot{R}_C dt+C\int_{t_{06}}^{t_{16}}\frac{d\tau_C}{dt}dt+\int_{t_{07}}^{t_{17}}\dot{R}_4 dt\right] \\ +\int_{t_{01}}^{t_{11}}\dot{R}_1 dt+C\int_{t_{02}}^{t_{12}}\frac{d\tau_1}{dt}dt+\int_{t_{03}}^{t_{13}}\dot{R}_2 dt+C\int_{t_{04}}^{t_{14}}\frac{d\tau_3}{dt}dt \\ +\int_{t_{05}}^{t_{15}}\dot{R}_3 dt+C\int_{t_{06}}^{t_{16}}\frac{d\tau_5}{dt}dt+\int_{t_{07}}^{t_{17}}\dot{R}_4 dt \tag{6.4 - 47}$$

This reduces to

$$\int_{t_{07}}^{t_{17}}K\dot{R}_S dt+\int_{t_{07}}^{t_{17}}\dot{R}_L dt=K[R_{1C}-R_{0C}+C(\tau_{1C}-\tau_{0C})+R_{14}-R_{04}] \\ +R_{11}-R_{01}+C(\tau_{11}-\tau_{01})+R_{12}-R_{02}+C(\tau_{13}-\tau_{03}) \\ +R_{13}-R_{03}+C(\tau_{15}-\tau_{05})+R_{14}-R_{04} \tag{6.4 - 48}$$

Rearranging terms and dividing by T

$$\frac{1}{T}\left[\int_{t_{07}}^{t_{17}} K\dot{R}_S dt + \int_{t_{07}}^{t_{17}} \dot{R}_L dt\right] = -\frac{C}{T}[KD_2 + D_4]$$
$$+\frac{(R_{14}+R_{13}+R_{12}+R_{12})-(R_{04}+R_{03}+R_{02}+R_{01})}{T} \tag{6.4 - 49}$$
$$+K\frac{(R_{14}+R_{1C})-(R_{04}+R_{0C})}{T}$$

which is similar to equation (6.4-45). Therefore,the integrated Doppler N_9 may be represented exactly as the time average of the range difference along the signal path (i.e., average range rate) where the weighting factor K is given by

$$K = \frac{f_Y}{f_X} \tag{6.4 - 50}$$

When modeled as an average range rate, the quantity considered as the observation is given by

$$\dot{\bar{R}}_{OBS} = -\frac{C}{2f_X}\left[\frac{N_8}{T} - f_B\right] \tag{6.4 - 51}$$

where
f_x = constant frequency determined by particular tracking configuration
f_B = ground generated bias frequency
N_8 = integrated Doppler plus bias cycles accumulated
T = integrated time interval required to accumulate N_8

6.4.4 Range Rate Partials

$$\dot{\bar{R}} = -\frac{C}{2f_X}\left[\frac{N_8}{T} - f_B\right] = \frac{\Delta R_L + \frac{f_Y}{f_X}\Delta R_S}{2T} \tag{6.4 - 52}$$

where

$$\Delta R_L = (R_{14} + R_{13} + R_{12} + R_{12}) - (R_{04} + R_{03} + R_{02} + R_{01})$$

$$\Delta R_S = (R_{14} + R_{1C}) - (R_{04} + R_{0C})$$

$$\frac{\partial \dot{\bar{R}}}{\partial X} = \frac{1}{2T}\left[\frac{\partial \Delta R_L}{\partial X} + \frac{f_Y}{f_X}\frac{\partial \Delta R_S}{\partial X}\right] \qquad (6.4 - 53)$$

$$\begin{aligned}\frac{\partial \Delta R_L}{\partial X_{TKNG_0}} = & \frac{\partial R_{14}}{\partial X_{TKNG}(t_{16})}\frac{\partial X_{TKNG}(t_{16})}{\partial X_{TKNG_0}} + \frac{\partial R_{13}}{\partial X_{TKNG}(t_{15})}\frac{\partial X_{TKNG}(t_{15})}{\partial X_{TKNG_0}} \\ & + \frac{\partial R_{12}}{\partial X_{TKNG}(t_{12})}\frac{\partial X_{TKNG}(t_{12})}{\partial X_{TKNG_0}} + \frac{\partial R_{11}}{\partial X_{TKNG}(t_{11})}\frac{\partial X_{TKNG}(t_{11})}{\partial X_{TKNG_0}} \\ & - \frac{\partial R_{04}}{\partial X_{TKNG}(t_{06})}\frac{\partial X_{TKNG}(t_{06})}{\partial X_{TKNG_0}} - \frac{\partial R_{03}}{\partial X_{TKNG}(t_{05})}\frac{\partial X_{TKNG}(t_{05})}{\partial X_{TKNG_0}} \\ & - \frac{\partial R_{02}}{\partial X_{TKNG}(t_{02})}\frac{\partial X_{TKNG}(t_{02})}{\partial X_{TKNG_0}} - \frac{\partial R_{01}}{\partial X_{TKNG}(t_{01})}\frac{\partial X_{TKNG}(t_{01})}{\partial X_{TKNG_0}}\end{aligned} \qquad (6.4 - 54)$$

$$\begin{aligned}\frac{\partial \Delta R_L}{\partial X_{TKED}} = & \frac{\partial R_{13}}{\partial X_{TKED}(t_{14})}\frac{\partial X_{TKED}(t_{14})}{\partial X_{TKED}} + \frac{\partial R_{12}}{\partial X_{TKED}(t_{13})}\frac{\partial X_{TKED}(t_{13})}{\partial X_{TKED}} \\ & - \frac{\partial R_{03}}{\partial X_{TKED}(t_{04})}\frac{\partial X_{TKED}(t_{04})}{\partial X_{TKED}} - \frac{\partial R_{12}}{\partial X_{TKED}(t_{03})}\frac{\partial X_{TKED}(t_{03})}{\partial X_{TKED}}\end{aligned} \qquad (6.4 - 55)$$

$$\begin{aligned}\frac{\partial \Delta R_S}{\partial X_{TKNG_0}} = & \frac{\partial R_{14}}{\partial X_{TKNG}(t_{16})}\frac{\partial X_{TKNG}(t_{16})}{\partial X_{TKNG}} + \frac{\partial R_{1C}}{\partial X_{TKNG}(t_{1C})}\frac{\partial X_{TKNG}(t_{1C})}{\partial X_{TKNG}} \\ & - \frac{\partial R_{04}}{\partial X_{TKNG}(t_{06})}\frac{\partial X_{TKNG}(t_{06})}{\partial X_{TKNG}} - \frac{\partial R_{0C}}{\partial X_{TKNG}(t_{0C})}\frac{\partial X_{TKNG}(t_{0C})}{\partial X_{TKNG}}\end{aligned}$$

(6.4 - 56)

$$\frac{\partial \Delta R_S}{\partial X_{TKED}} = 0 \qquad (6.4 - 57)$$

The problem of satellite-to-satellite tracking is discussed in greater detail in Reference [6-3] of this section.

6.5 PCE MEASUREMENT TYPES

The PCE measurement types are sets of elements precisely determined in previous GEODYN orbit declination runs.

The inertial Cartesian elements obtained from interpolation of the integrator output are used as the calculated measurements for PCE types, $x, y, z, \dot{x}, \dot{y}, \dot{z}$.

The partials of these measurements are

$$\begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} & \frac{\partial x}{\partial \dot{x}} & \frac{\partial x}{\partial \dot{y}} & \frac{\partial x}{\partial \dot{z}} \\ \frac{\partial y}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} & \frac{\partial y}{\partial \dot{x}} & \frac{\partial y}{\partial \dot{y}} & \frac{\partial y}{\partial \dot{z}} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial \dot{x}} & \frac{\partial z}{\partial \dot{y}} & \frac{\partial z}{\partial \dot{z}} \\ \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \dot{y}} & \frac{\partial \dot{x}}{\partial \dot{z}} \\ \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} & \frac{\partial \dot{y}}{\partial \dot{x}} & \frac{\partial \dot{y}}{\partial \dot{y}} & \frac{\partial \dot{y}}{\partial \dot{z}} \\ \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \dot{x}} & \frac{\partial \dot{z}}{\partial \dot{y}} & \frac{\partial \dot{z}}{\partial \dot{z}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6.5 - 1)$$

The osculating elements obtained by conversion of the above mentioned Cartesian elements are used as the calculated measurements for PCE types, $a, e, i, \Omega, \omega, M$.

Thee partials for these measurements are given in Section 11.4.

6.6 AVERAGE RANGE RATE MEASUREMENT TYPES

Figure 6.6-1 illustrates the geometry of the average range rate measurement types. A signal is transmitted from a transmitter to a satellite, then a ground station receives the signal from the satellite, and
ρ_T - is the transmitter-satellite range
ρ_R - is the satellite-receiver range
$\bar{R}_R$ - is the position vector of the receiver
$\bar{R}_T$ - is the position vector of the transmitter
$\bar{R}_S$ - is the position vector of the satellite

If t_6 is the recorded time of the end of the doppler counting interval at the receiver, and if T is the length Of the counting interval, then the average range rate measurement is

$$\dot{\rho} = \frac{\rho_R(t_6, t_5) + \rho_T(t_5, t_4) - \rho_R(t_3, t_2) - \rho_T(t_2, t_1)}{2T} \qquad (6.6 - 1)$$

Where it is necessary to iterate for the satellite and transmitter times,

$$
\begin{aligned}
t_5 &= t_6 - \frac{\rho_R(t_6, t_5)}{c} \\
t_4 &= t_5 - \frac{\rho_T(t_5, t_4)}{c} \\
t_3 &= t_6 - T \\
t_2 &= t_3 - \frac{\rho_R(t_3, t_2)}{c} \\
t_1 &= t_2 - \frac{\rho_T(t_2, t_1)}{c}
\end{aligned}
$$

and where

$$
\begin{aligned}
\rho_R(t_6, t_5) &= |\bar{R}_R(t_6) - \bar{R}_S(t_5)| \\
\rho_R(t_5, t_4) &= |\bar{R}_T(t_4) - \bar{R}_S(t_5)| \\
\rho_R(t_3, t_2) &= |\bar{R}_R(t_3) - \bar{R}_S(t_2)| \\
\rho_R(t_2, t_1) &= |\bar{R}_T(t_1) - \bar{R}_S(t_2)|
\end{aligned} \qquad (6.6 - 2)
$$

A two-way average range rate measurement is a special case of the three-way average range rate measurement. For the two way average range rate measurement, the receiver and the transmitter are the same. Therefore,

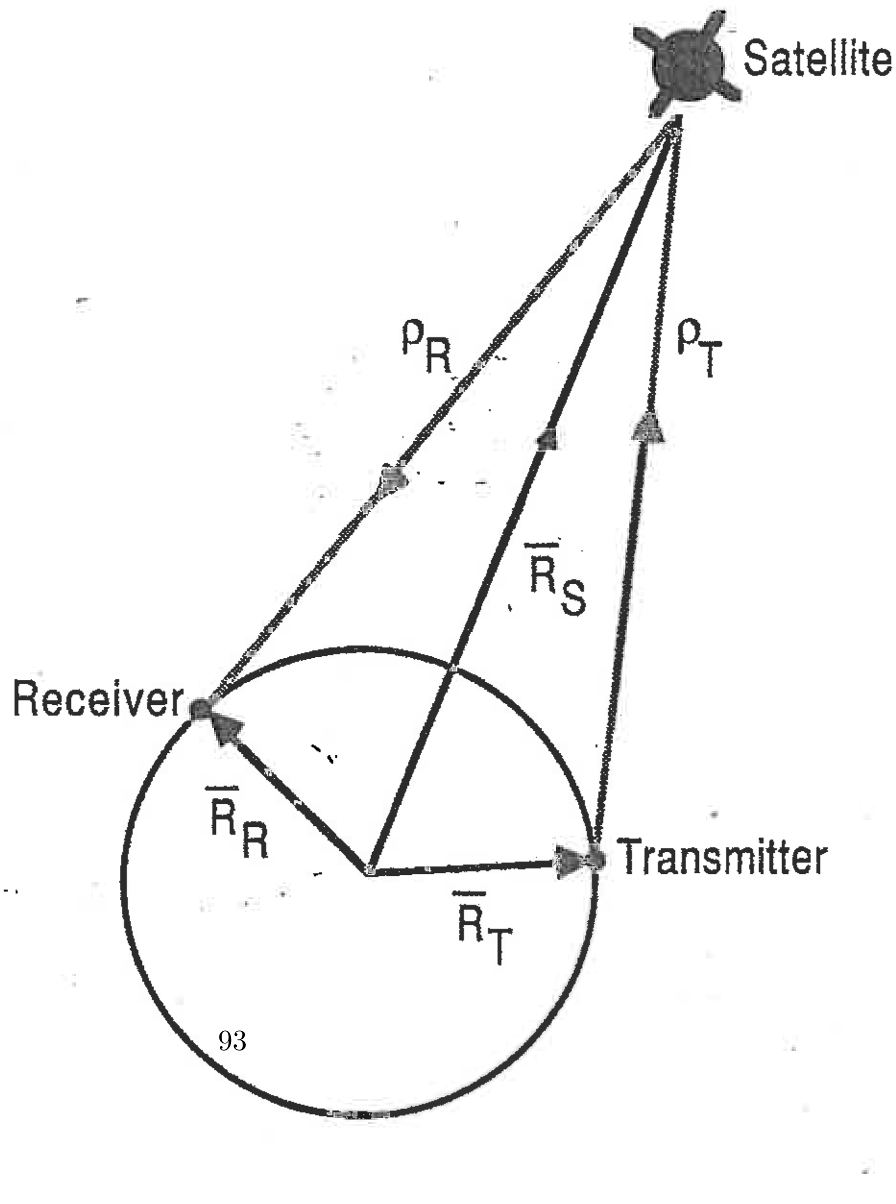
$$
\begin{aligned}
\rho_T &= \rho_R \\
\bar{R}_T &= \bar{R}_T
\end{aligned}
$$

The Partial Derivatives are:

$$
\frac{\partial \dot{\bar{\rho}}}{\partial r_i} = \frac{1}{2T} \left[\frac{\partial \rho_R(t_6, 6_5)}{\partial r_i} + \frac{\partial \rho_T(t_5, t_4)}{\partial r_i} - \frac{\partial \rho_R(t_3, t_2)}{\partial r_i} - \frac{\partial \rho_T(t_2, t_1)}{\partial r_i} \right]
$$

where the partial $\frac{\partial \rho}{\partial r_i}$is given in Section 6.2.

Figure 11. Figure 6.6-1 Geometry for Average Range Rate Measurement



6.7 THE 8906 OCEAN TIDE ALTIMETRIC MEASUREMENT CORRECTION

Previously, a measurement correction was applied based upon the Parke model, using only sub-satellite latitude and longitude as input. The correction was either on or off, with no input parameter control. No partials were generated. The new tidal measurement model, similar to the force model discussed in Section 8.8.3, is based on the following correction:

$$\Delta h = \sum_f \sum_{lq\pm} (1 + \alpha_l h_l') \frac{V_f(t)}{\bar{V}_f} C^{\pm}_{lqf} \bar{P}_{lq}(\sin\phi) \cos\alpha^{\pm}_{lqf} \tag{6.7 - 1}$$

where

$$\begin{aligned} \alpha^{\pm}_{lqf} = (\pm)[(2-2h)\omega^* + (2-2h+j)M^* + k\Omega] \\ +m\Theta_g \pm q\lambda + \pi - m\frac{\pi}{2} + \zeta^{\pm}_{lqf} \end{aligned} \tag{6.7 - 2}$$

and where:

$$\alpha_l = \frac{3\rho_0}{\rho_e(2l+1)}$$

and where:

$$V_f = G_D A_f$$

and where:

ϕ, λ, r latitude, east longitude and radial distance of the point of evaluation;

Θ_g Greenwich sidereal hour angle;

$\bar{P}_{lq}(\sin\phi)$ associated normalized Legendre functions;

$\alpha^*, e^*, i^*, \Omega^*, \omega^*, M^*$ Keplerian elements of the disturbing body referred to ecliptic;

G_D equivalent to Doodson constant;

$\sum_f$ summation over all tide constituents (f) in the expression of the tide-generating potential;

A_f equivalent to Doodson coefficient;

h_t' load deformation coefficients;

ρ_0 average density of the oceanic water;

C_{lqf} amplitude and phase in the spherical harmonic expansion of the ocean tides specified by $l, q, \pm$, and the tide constituent f

ρ_e average density of the earth.

This model is a function of the amplitude and phase information supplied on the OTIDES and OTIDEN cards. It is now possible to use altimetry measurement partials to contribute more information to the solution of the input ocean tide parameters. The code for equations (6.7-1) and (6.7-2) is primarily located in subroutine ALTOTD. Once per global iteration initializations are performed in PRETID and PRESET. The Doodson coefficient is computed in ADOOD. Please see Section 8.8.3 for further details.

6.8 PLANETARY EPHEMERIS CORRECTION

The precomputed ephemerides may be differentially corrected with conic formulae as described in [6-5]. Position and velocity are interpolated from the JPL ephemeris at an epoch of osculation specified by the user and are converted to orbital elements (Brouwer and Clemence Set III). The elliptical orbit with these elements agrees with the ephemeris orbit exactly at the osculation epoch and approximately at other epochs. Solve-for parameters are corrections to the orbital elements of the ephemeris at osculation epoch T_0. Partial derivatives of position and velocity with respect to these orbital elements are approximated by those elements from the osculating elliptical orbit. These partial derivatives are used to determine corrections in the osculating orbital elements, and from those, to apply linear differential correction to the ephemeris. The parameters that may be adjusted or provided as input via the global option card PLNBPH are any of the following:

$$\Delta E = \begin{bmatrix} \frac{\Delta a}{a} \\ \Delta e \\ \Delta M_o + \Delta w \\ \Delta \bar{p} \\ \Delta \bar{q} \\ e\Delta \bar{w} \end{bmatrix} \qquad (6.8 - 1)$$

where:

a - semi-major axis

e - eccentricity

M_o - Mean anomaly at osc. epoch T_o (ET)

$\Delta\bar{p}, \Delta\bar{q}, \Delta\bar{w}$ rotations about axis $\bar{P}, \bar{Q}, \bar{W}$ axes respectively, where $\bar{P}$ is the direction from focus to perifocus, $\bar{Q}$ is $\frac{\pi}{2}$ ahead of $\bar{P}$ in the orbital plane and $\bar{W} = \bar{P} \times \bar{Q}$.

Given position and velocity $\bar{r}$ and $\dot{\bar{r}}$ obtained from a planetary ephemeris at any time T, then the corrected position and velocity for the nth global iteration are computed from:

$$\bar{r}_n = \bar{r} + \frac{\partial \bar{r}}{\partial E}\Delta E_n \qquad (6.8 - 2)$$

$$\frac{\partial \bar{r}}{\partial E} = \left[\frac{\partial \bar{r}}{\partial \frac{\Delta a}{a}}, \frac{\partial \bar{r}}{\partial \Delta e}, \frac{\partial \bar{r}}{\partial(\Delta M_o + \Delta w)}, \frac{\partial \bar{r}}{\partial \Delta \bar{p}}, \frac{\partial r}{\partial \Delta \bar{q}}, \frac{\partial \bar{r}}{\partial e \Delta \bar{w}}\right] \qquad (6.8 - 3)$$

$$\frac{\partial \bar{r}}{\partial E_i} = \left[\frac{\partial x}{\partial E_i}\frac{\partial y}{\partial E_i}\frac{\partial z}{\partial E_i}\right]T \qquad (6.8 - 4)$$

$$\dot{\bar{r}}_n = \dot{\bar{r}} + \frac{\partial \dot{\bar{r}}}{\partial E}\Delta E_n \qquad (6.8 - 5)$$

$$\frac{\partial \dot{\bar{r}}}{\partial E_o} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial E_i} \\ \frac{\partial \dot{y}}{\partial E_i} \\ \frac{\partial \dot{z}}{\partial E_i} \end{bmatrix} \qquad (6.8 - 6)$$

First the osculating epoch T_0 is designated through user input on the cards PLNEPH. If it is omitted GEODYN will select the start of run. Then the planet's state vector at T_0 is calculated from the ephemeris as:

$\bar{r}_0$ = 12000.0 position Interpolated from ephemerides at osculation epoch T_0

$\dot{\bar{r}}_0$ = J2000.0 velocity interpolated from ephemerides at osculation epoch T_0

Given below are formulae to convert $\bar{r}_0$ and $\dot{\bar{r}}_0$ to the elliptical orbital elements E at T_0 [6 - 5] and [6 - 6], where the parameter μ is computed from:

$$\mu(planet) = \mu_s + \mu_p \qquad (6.8 - 7)$$

where μ_s and μ_p are the gravitational constants for the sun and the planet respectively.

$$r_0 = (\bar{r}_0 \cdot \bar{r}_0)^{\frac{1}{2}} \qquad (6.8 - 8)$$

The semimajor axis a is given by

$$a = \frac{1}{\frac{2}{r_0} - \frac{\dot{\bar{r}}_0 \cdot \dot{\bar{r}}_0}{\mu}} \qquad (6.8 - 9)$$

The mean motion η is computed from

$$\eta = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}} \qquad (6.8 - 10)$$

and the following computations are made:

$$e\cos E_0 = 1 - \frac{r_o}{a} \qquad (6.8 - 11)$$

$$e\sin E - 0 = \frac{\bar{r} \cdot \dot{\bar{r}}_0}{(\mu\alpha)^{\frac{1}{2}}} \qquad (6.8 - 12)$$

$$e = [(e\cos E_0)^2 + (e\sin E_0)^2]^{\frac{1}{2}} \qquad (6.8 - 13)$$

$$\cos E_0 = \frac{(e\cos E_0)}{e} \qquad (6.8 - 14)$$

$$\sin E_0 = \frac{(e\sin E_0)}{e} \qquad (6.8 - 15)$$

The unit vectors $\bar{P}$,$\bar{Q}$ and $\bar{W}$ are computed from

$$\bar{P} = \frac{\cos E_0}{r_0}\bar{r}_0 - (\frac{a}{\mu})^{\frac{1}{2}} \sin E_0 \dot{\bar{r}}_0 \qquad (6.8 - 16)$$

$$\bar{W} = \frac{\bar{r}_0 \times \dot{\bar{r}}_0}{[\mu a(1-e^2)]^{\frac{1}{2}}} \qquad (6.8 - 17)$$

$$\bar{Q} = \bar{W} \times \bar{P} \qquad (6.8 - 18)$$

The partial derivatives $\frac{\partial \bar{r}}{\partial E}$ and $\frac{\partial \dot{\bar{r}}}{\partial E}$ are computed from the orbital elements a, e, η, $\bar{P}$,$\bar{Q}$ and $\bar{W}$ which are computed once at epoch T_0 and from the following quantities, which are computed at each time T that the partials are evaluated.
$\bar{r}$,$\dot{\bar{r}}$ = 12000.0 position and velocity interpolated from the ephemeris at time T (ephemeris time)

$$r = (\bar{r} \cdot \bar{r})^{\frac{1}{2}} \qquad (6.8 - 19)$$

$$r\dot{r} = \bar{r} \cdot \dot{\bar{r}} \qquad (6.8 - 20)$$

$$\dot{r} = \frac{(r\dot{r})}{r} \qquad (6.8 - 21)$$

$$\ddot{\bar{r}} = \frac{\mu}{r^3}\bar{r} \qquad (6.8 - 22)$$

The partial derivatives of $\bar{r}$ at ephemeris time T with respect to each element of ΔE are given by

$$\frac{\partial \bar{r}}{\partial(\frac{\Delta a}{a})} = \bar{r} - \frac{3}{2}(T - T_0)\dot{\bar{r}} \tag{6.8 - 23}$$

$$\frac{\partial \bar{r}}{\partial(\Delta e)} = H_1 \bar{r} + K_1 \dot{\bar{r}} \tag{6.8 - 24}$$

where the quantities H_1 and K_1 which are functions of T, are given by

$$H_1 = \frac{r - a(1+e^2)}{ae(1-e^2)} \tag{6.8 - 25}$$

$$K_1 = \frac{r\dot{r}}{a^2\eta^2 e}\left[1 + \frac{r}{a(1-e^2)}\right] \tag{6.8 - 26}$$

$$\frac{\partial \bar{r}}{\partial(\Delta M_o + \Delta w)} = \frac{\dot{\bar{r}}}{\eta} \tag{6.8 - 27}$$

$$\frac{\partial \bar{r}}{\partial(\Delta \bar{p})} = \bar{P} \times \bar{r} \tag{6.8 - 28}$$

$$\frac{\partial \bar{r}}{\partial(\Delta \bar{q})} = \bar{Q} \times \bar{r} \tag{6.8 - 29}$$

$$\frac{\partial \bar{r}}{\partial(e\Delta \bar{w})} = \frac{1}{e}(\bar{W} \times \bar{r} - \frac{\dot{\bar{r}}}{\eta}) \tag{6.8 - 30}$$

Differentiating Eqs, [6.8-23] - [6.8 - 30] with respect to ephemeris time gives the partial derivatives of $\bar{r}$ at ephemeris time t with respect to each element of ΔE: .

$$\frac{\partial \dot{\bar{r}}}{\partial(\frac{\Delta a}{a})} = -\frac{1}{2}\dot{\bar{r}} - \frac{3}{2}(T - T_0)\ddot{\bar{r}} \tag{6.8 - 31}$$

$$\frac{\partial \dot{\bar{r}}}{\partial(\Delta e)} = H_2 \bar{r} + K_2 \dot{\bar{r}} \tag{6.8 - 32}$$

where

$$H_2 = \frac{\dot{r}}{ae(1-e^2)}\left\{1 - \frac{a}{r}[1 + \frac{a}{r}(1-e^2)]\right\} \tag{6.8 - 33}$$

$$K_2 = \frac{1}{e(1-e^2)}(1 - \frac{r}{a}) \tag{6.8 - 34}$$

$$\frac{\partial \dot{\bar{r}}}{\partial(\Delta M_o + \Delta w)} = \frac{\ddot{\bar{r}}}{\eta} \tag{6.8 - 35}$$

$$\frac{\partial \dot{\bar{r}}}{\partial(\Delta \bar{p})} = \bar{P} \times \dot{\bar{r}} \tag{6.8 - 36}$$

$$\frac{\partial \dot{\bar{r}}}{\partial(\Delta \bar{q})} = \bar{Q} \times \dot{\bar{r}} \tag{6.8 - 37}$$

$$\frac{\partial \dot{\bar{r}}}{\partial(e\Delta \bar{w})} = \frac{1}{e}(\bar{W} \times \dot{\bar{r}} - \frac{\ddot{\bar{r}}}{\eta}) \tag{6.8 - 38}$$

6.9 TABLE COMPRESSION VIA 2D FOURIER TRANSFORMS

6.9.1 Introduction

As described in the previous section, range correction tables were generated at 17 incidence angles representing 21 different laser station hardware configurations/characteristics in use today. Stations with similar configurations arc grouped as shown in Geodyn II File Description Volume 5, Section 2.9. Each table is a 51x51 matrix of 2 micro-radian resolution pixels reproducing the far field diffraction pattern (FFDP) at each incidence angle. A data compression technique was employed to limit the overall size and computer memory storage requirements of the resulting TOPEX/Poseidon LRA range correction tables. Specifically, a two-dimensional Fourier transform representation of the tables was investigated and ultimately adopted.

6.9.2 Theoretical Basis

The data compression goal was to duplicate the original data to within 5mm at any point within the table. Initially, the range correction tables were expressed in polar coordinates for input into the Fourier transform. However, in this representation, the range correction values were discontinuous at the tables boundaries and, therefore, difficult to fit with a periodic function. Adoption of cartesian coordinates minimized this effect. The following formulation was used:

$$f(X,Y) = a + b \cdot X + c \cdot Y + \sum_{n=1}^{N} d + n1 \cos(s) + f_{n1} \sin(s) + \sum_{m=1}^{M} d_{m1} \cos(t) +$$
$$e_{1m} \sin(t) + \sum_{n=2}^{N} \sum_{m=2}^{M} d_{mn} \cos(s)\cos(t) + e_{nm} \cos(s)\sin(t) +$$
$$f_{nm} \sin(s)\cos(t) + g_{nm} \sin(s)\sin(t) \quad (6.9 - 1)$$

where

$$s = (n-1)2\Pi \frac{X - X_{min}}{Y_{max} - Y_{min}} \quad (6.9 - 2)$$
$$t = (n-1)2\Pi \frac{Y - Y_{min}}{Y_{max} - Y_{min}} \quad (6.9 - 3)$$

A least squares was used to obtain the coefficient values for the Fourier series needed for each table. TOPEX/Poseidon orbits the Earth at an altitude of 1334 km and its velocity aberration is limited to an approximate range of 20 to 50 micro-radians. The Fourier transform fit to the range correction data was improved by eliminating all points physically outside this region. After a series of tests it was determined that a maximum degree of 5 (N = M = 5) for both the X and Y directions provided the best balance between the number of Fourier coefficients and the fit to the data. This reduced the number of values stored for each table from 2,600 gridded values to 83 Fourier coefficients.

6.9.3 Application of tile Range Correction Algorithms

The Fourier coefficient representation of the LRA range correction has been implemented directly in the GEODYN orbit determination code [6-8]. Figure 6.9.1 defines a right-handed coordinate system whose origin is at the center of the Earth [6-7]. The Z_0 axis points through the laser tracking station. The X_0 axis is defined as the cross product of the satellite position vector and the station vector. The Y_0 points toward the horizon in the plane containing the spacecraft and completes the right-handed system. Note that the satellite and station are both in the YZ plane and that the subsequent calculation assumes a spherical orbit. The rectangles demonstrate the orientation of the far-field diffraction pattern (FFDP) in this coordinate system. Plane P contains the satellite velocity vector and is normal to the satellite position vector (spherical orbit). Angle ω is the azimuth of the satellite velocity with respect to the $-X_0$ axis. Plane P', the plane of the diffraction

pattern, is normal to the transmitter-satellite line of sight. The V' vector is the projection of the satellite velocity vector in the P' plane and is separated from the $-X_0$ axis by an angle η. Angle θ_L separates planes P and P'.
The location of range correction on the grid is computed using the following equations

$$\begin{aligned} x &= \Psi \cos \eta \\ y &= \Psi \sin \eta \end{aligned} \tag{6.9 - 4}$$

where

$$\Psi = \frac{2V'}{c}$$

$$\begin{aligned} \eta &= \tan^{-1}(\cos \theta_L \tan \omega) \\ &= ATAN2(\sin \omega \cos \theta_L \cos \omega) \end{aligned} \tag{6.9 - 5}$$

where

$$V = \sqrt{\frac{R_e g}{|\vec{r}|}} \tag{6.9 - 6}$$

$$V' = V \sqrt{\cos^2 \omega + \cos^2 \theta_L \sin^2 \omega}$$

$$\theta_L = \cos^{-1} \sqrt{1 - \left(\frac{R_e}{|\vec{r}|}\right)^2 \sin^2 \theta_s} \tag{6.9 - 7}$$

$$\omega = \cos^{-1}(-\vec{X}_0 \cdot \frac{\vec{V}}{|\vec{V}|}) \tag{6.9 - 8}$$

$$\vec{X}_0 = \frac{\vec{P}}{|\vec{p}|} \times \frac{\vec{r}}{|\vec{r}|} \tag{6.9 - 9}$$

and where
Ψ = magnitude of displacement from center of grid
n = azimuth angle of V' measured from $-x_0$ in plane P'
V'= component of satellite velocity normal to line of sight
ω = azimuth angle of V measured from $-x_0$ in plane P
θ_s= zenith angle
R_e = Earth's radius
g = gravitationaI constant
c = speed of light
p = line of sight vector from station to satellite
r = satellite position vector

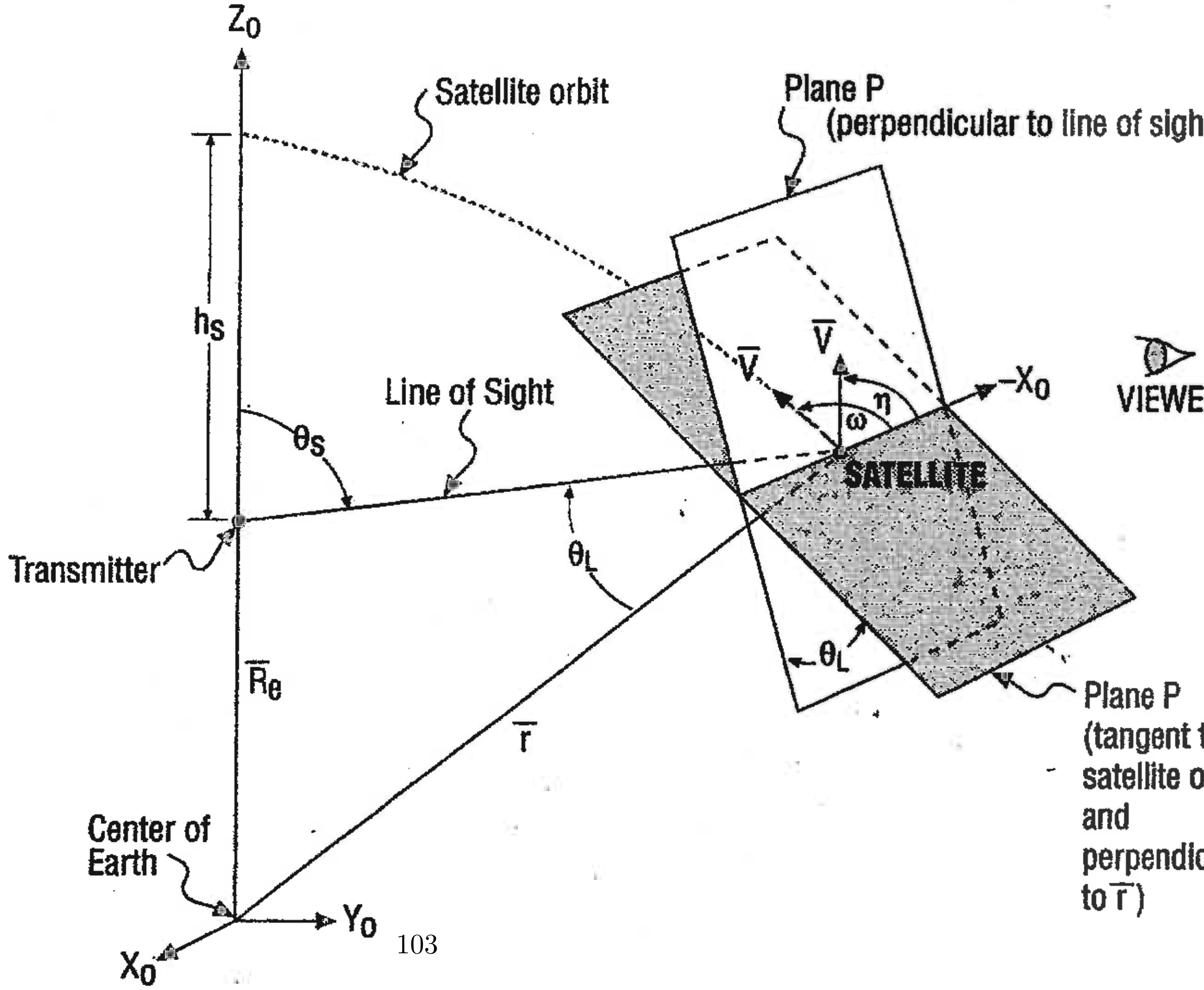
For any given observation, the range correction is linearly interpolated between the function values bounding the incidence angle. If the observation incidence angle exceeds the maximum value in the tables, nominally 60 degrees, the maximum value is used without interpolation.

Figure 12. Figure 6.9-1 Right handed coordinate system: whose origin is at the Center of the Earth.

$\vec{v}$ lies in plane P perpendicular to $\vec{r}$ and at an angle ω from the $-x_0$ direction measured in P.

$\vec{v}$ is the projection of $\vec{v}$ onto the plane P perpendicular to the line of sight, and at and angle η from the $-x_0$ direction measured in P.

The view is from beyond the satellite orbit.



$\vec{v}$ lies in plane P perpendicular to $\vec{r}$ and at an angle ω from the $-x_0$ direction measured in P.

$\vec{v}$ is the projection of $\vec{v}$ onto the plane P perpendicular to the line of sight, and at and angle n from the $-x_0$ direction measured in P. The view is from beyond the satellite orbit.

7 DATA PREPROCESSING

The function of data preprocessing is to convert and correct the data. These corrections and conversions relate the data to the physical model, and to the coordinate and time reference systems used in GEODYN. The data corrections and conversions implemented in GEODYN are to transform all observation times to ET time at the satellite refer right ascension and declination observations to the true equator and equinox of date correct range measurements for transponder delay and gating effects correct right ascension and declination observations for annual and diurnal aberration correct for refraction and parallactic refraction convert TRANET Doppler observations into range rate measurements. Some of these conversions and corrections are performed 'when the data is reformatted, others are applied to the data on the first iteration of each arc. Each of these preprocessing items is considered in greater detail in the subsections which follow.

7.1 TIME PREPROCESSING

The time reference system used to specify the time of each observation is determined by a time identifier on the data record. This identifier also specifies whether the time recorded was the time at the satellite or at the observing station. The preprocessing ill GEODYN.transforms all observations to ET time when the data 1S reformatted. If the time recorded is the time at the station, it is converted to time at the satellite. This conversion is applied using a correction equal to the propagation time between the spacecraft and the observing station. The station-satellite distance used for this correction is computed from the initial estimate of the trajectory. There is special preprocessing for right ascension and declination measurements from the GEOS satellites when input in National Space Science Data Center format. If the observation is passive, the image recorded is an observation of light reflected from the satellite and the times are adjusted for propagation delay as above. If the observation is active, the image recorded is an observation of light transmitted from the optical beacon on the satellite. The beacons on the GEOS satellites were programmed to produce a sequence of seven flashes at four second intervals starting on an even minute. For the active observations, the times are set equal to the programmed flash time with a correction applied for known clock errors (Reference I). plus half a millisecond, the time allowed for flash buildup. The clock corrections for the active observations are computed by simple linear interpolation in a table of known errors in the satellite on-board clock.

7.2 REFERENCE SYSTEM CONVERSION TO TRUE OF DATE

The camera observations, right ascension and declination, may be input referred to the mean equator and equinox of date, to the true equator and equinox of date, or to the mean equator and equinox of some standard epoch. The GEODYN system transforms these observations to the true equator and equinox of date.

7.3 TRANSPONDER DELAY AND GATING EFFECTS

The range observations may be corrected for transponder delay or gating errors. If requested, GEODYN applies the corrections. The transponder delay correction is computed as a polynomial in the range rate:

$$\Delta\rho = a_0 + a_1\dot{\rho} + a_2(\dot{\rho})^2 \qquad (7.3\text{- }1)$$

where a_0, a_1 and a_2 depend on the characteristics of the particular satellite.
A gating error is due to the fact that actual range measurements are either time delays between transmitted and received radar pulses or the phase shifts in the modulation of a received signal with respect to a coherent transmitted signal. Thus there is the possibility of incorrectly identifying the returned pulse or the number of integral phase shifts. The difference between the observed range and the computed range on the first iteration of the arc is used to determine the appropriate correction. The correction is such that there is less than half a gate, where the gate is the range equivalent of the pulse spacing or phase shift. The appropriate gate of course depends on the particular station.

7.4 ABERRATION

Optical measurements may require corrections (Reference 2) for the effects of annual aberration and diurnal aberration.

Annual aberration
The corrections to right ascension and declination measurements far annual aberration

effects are given by

$$\alpha = \alpha' - \frac{20".5(\cos\alpha' \cos\omega \cos\epsilon_T + \sin\alpha' \sin\omega)}{\cos\delta'}$$
$$\delta = \delta' - 20".5[\cos\omega \cos\epsilon_T(\tan\epsilon_T \cos\delta' - \sin\alpha' \sin\delta') + \cos\alpha' \sin\delta' \sin\omega]$$

where

α - true right ascension of the satellite

α' - observed right ascension of the satellite

δ - true declination of the satellite

δ' - observed declination of the satellite

ϵ_T - true obliquity of date

ω - geocentric longitude of the sun in the ecliptic plane

Diurnal Aberration
The corrections to right ascension and declination measurements for diurnal aberration effects are given by

$$\alpha = \alpha' + 0".320 \cos\phi' \cos h_s \sec\delta'$$
$$\delta = \delta' + 0".320 \cos\phi' \sin h_x \sin\phi'$$

where

ϕ' - geocentric latitude of the station

h_s - local hour angle measured in the westward direction from the station to the satellite

α - true right ascension of the satellite

α' - observed right ascension of the satellite

δ - true declination of the satellite

δ' - observed declination of the satellite

7.5 REFRACTION CORRECTIONS

The GEODYN system can apply corrections to all of the observational types significantly affected by refraction.

Parallatic Refraction for Right Ascension and Declination

Optical measurements may require corrections (References 3, 4, 5) for the effects of parallactic refraction. These corrections are given by

$$\alpha = \alpha' - \Delta R \frac{\sin q}{\cos \delta}$$
$$\delta = \delta' - \Delta R \cos q$$

where the change in the zenith angle, ΔR , in radians is given by

$$\Delta R = -\frac{0.453(4.84813)\tan Z_0}{\rho \cos Z_0}[1 - e^{(-1.385)}10^{-4}\rho \cos Z_0] \quad (7.5\ \text{-}1)$$

and

α - true right ascension of the satellite

α' - observed right ascension of the satellite

δ - true declination of the satellite

δ' - observed declination of the satellite

Z_0 - observed zenith angle in radians

ρ - range from the station to the satellite in meters

q - parallactic angle in radians

The parallactic angle q is defined by the intersection of two planes represented by their normal vectors $\bar{P}_1$ and $\bar{P}_2$.

$$\bar{P}_1 = \hat{C_P} \times \hat{u}$$
$$\bar{P}_2 = \hat{V} \times \hat{u}$$

where

$\hat{C_P}$ = (0,0,1)

$\hat{V}$ is the local vertical at the station (defined in inertial coordinates)

$\hat{u}$ is the unit vector pointing from the station to the satellite in inertial space

Therefore, the sine and cosine of the parallactic angle are given by

$$\cos q = \hat{P}_1 \cdot \hat{P}_2$$
$$\sin q = \hat{P}_3 \cdot \hat{P}_2$$

where

$\hat{P}_1$ is the unit vector in the $\bar{P}_1$ direction

$\hat{P}_2$ is the unit vector in the $\bar{P}_2$ direction

and

$$\hat{P}_3 = \frac{\bar{P}_1 \times \hat{u}}{|\bar{P}_1 \times \hat{u}|}$$

The parallactic angle, q , is measured in the Clockwise direction about the station-satellite vector (i.e., a left-handed system is used to define this angle). All vectors and vector cross products used in this formulation conform to a right-handed system.

Tropospheric Refraction Correction

The tropospheric refraction correction model used by GEODYN (for all measurement types with the exception of right ascension and declination) is the Hopfield Model modified by Goad to use the Saastamoinen zenith range corrections. [Reference 6]

The Sasstamoinen zenith range correction is used to determine the height of the troposphere appropriate for the wet and dry components.

$$h_1 = h_{dry} = \frac{5.0 \times 0.002277}{N_1 \times 10^{-6}} P \tag{7.5 -2}$$

where

h_{dry} = dry tropospheric height in meters

P = surface pressure in millibars (mb)

N_1 = deviation from unity of the surface index of refraction (dry component)

$= N_{dry} = 77.624 \times 10^{-6} \frac{P}{T}$

$h_{wet} = \frac{5.0 \times 0.002277}{N_2 \times 10^{-6}} [\frac{125}{T} + 0.05]$e

where

h_{wet} = wet tropospheric height in meters

T = surface temperature in degrees K

e = partial pressure of water vapor in milibars (rob)

N_2 = deviation from unity of the surface index of refraction(wet component)

= h_{wet} =371900 $\times 10^{-6}$ $\frac{e}{T^2}$ - 12.92 $\times 10^{-6} \frac{e}{T}$

Range Correction

$$\Delta R_{meters} = C\left[\frac{N_1}{10^6}\sum_{i=1}^{9}\frac{\alpha_{i1}r_1^i}{i} + \frac{N_2}{10^6}\sum_{i=1}^{9}\frac{\alpha_{i2}r_2^i}{i}\right] \qquad (7.5\text{ -3})$$

where

C = $\left[\frac{170.2649}{173.3-\frac{1}{\lambda^2}}\right]\left[\frac{78.8828}{77.624}\right]\left[\frac{173.3+\frac{1}{\lambda^2}}{173.3-\frac{1}{\lambda^2}}\right]$

$\simeq$ 1 for radio frequencies

λ = wavelength of transmission in microns

$r_j = \sqrt{(\alpha_e + h_j)^2 - (\alpha_e \cos\theta)^2} - \alpha_e \sin\theta$

= range to top of wet (j = 2) or dry (j = 1) component

and

α_e is semi-major axis of the earth is the elevation angle described above

θ is the elevation angle

N_j, h_j described above

α_{ij} defined as follows

$\alpha_{1j} = 1$

$\alpha_{2j} = 4a_j$

$\alpha_{3j} = 6a_j^2 + 4b_j$

$$
\begin{aligned}
\alpha_{4j} &= 4a_j(a_j^2 + 3b_j) \\
\alpha_{5j} &= a_j^4 + 12a_j^2 b_j + 6b_j^2 \\
\alpha_{6j} &= 4a_j b_j(a_j^2 + 3b_j) \\
\alpha_{7j} &= b_j^2(6a_j^2 + 4b_j) \\
\alpha_{8j} &= 4a_j b_j^3 \\
\alpha_{9j} &= b_j^4
\end{aligned}
$$

and

$$
\begin{aligned}
a_j &= -\tfrac{\sin\theta}{h_j} \\
b_j &= -\tfrac{\cos^2\theta}{2h_j a_e}
\end{aligned}
$$

where all j subscripts denote the following:
$j = 1$ wet component
$j = 2$ dry component

Range Rate Correction

$$
\Delta\dot{R}_{meters} = C\Bigg[\sum_{j=1}^{2} \frac{N_j}{10^6}\Bigg[\frac{\partial r_j}{\partial\theta}\sum_{i=1}^{9}\alpha_{ij} r_j^{i-1} + \frac{\partial a_j}{\partial\theta}\sum_{i=1}^{9}\frac{\beta_{ij} r_j^i}{i} + \frac{\partial b_j}{\partial\theta}\sum_{i=1}^{9}\frac{\gamma_{ij} r_j^i}{i}\Bigg]\Bigg] \qquad (7.5\text{ -4})
$$

where

N_j, α_{ij}, C, r_j described above

$$
\begin{aligned}
\tfrac{\partial r_j}{\partial\theta} &= \left[a_e\cos\theta - \tfrac{\alpha_e^2\cos\theta\sin\theta}{\sqrt{(\alpha_e+h_j)^2-\alpha_e^2\cos^2\theta}}\right]\dot{\theta} \\
\tfrac{\partial a_j}{\partial\theta} &= -\tfrac{\cos\theta}{h_j}\dot{\theta} \\
\tfrac{\partial b_j}{\partial\theta} &= \tfrac{\cos\theta\sin\theta}{a_e h_j}\dot{\theta} \\
\beta_{ij} &= \tfrac{\partial\alpha_{ij}}{\partial\alpha_j}
\end{aligned}
$$

and

$$
\gamma_{ij} = \tfrac{\partial\alpha_{ij}}{\partial b_j}
$$

so

$$
\begin{aligned}
\gamma_{1j} &= 0 \\
\beta_{1j} &= 0 = \gamma_{2j} \\
\beta_{2j} &= 4 = \gamma_{3j} \\
\beta_{3j} &= 12a_j = \gamma_4 j \\
\beta_{4j} &= 12a_j^2 + 12b_j = \gamma_{5j} \\
\beta_{5j} &= 4a_j^3 b_j + 24a_j b_j = \gamma_{6j} \\
\beta_{6j} &= 12a_j^2 b_j + 12b_j^2 = \gamma_{7j} \\
\beta_{7j} &= 12a_j b_j^2 = \gamma_{8j} \\
\beta_{8j} &= 4b_j^3 = \gamma_{9j} \\
\beta_{9j} &= 0
\end{aligned}
$$

where all j subscripts denote the following:
$j = 1$ wet component
$j = 2$ dry component

Elevation Angle Corrections

$$
\Delta\theta = C_0 \frac{4\cos\theta}{r_{sat}} \left[\sum_{j=1}^{2} \frac{N_j}{h_j} \left[\frac{\sum_{i=1}^{7} \chi_{ij} r_j^{i+1}}{i(i+1)} + \Delta\theta_j (r_{sat} - r_j) \right] \right] \tag{7.5 -5}
$$

where

$$
\begin{aligned}
C_0 &= \left[\frac{170.2649}{173.3 - \frac{1}{\lambda^2}} \right] \left[\frac{78.8828}{77.624} \right] \\
r_{sat} &= \text{range to the satellite} \\
N_j, h_j, r_j and \theta &\ \text{defined above} \\
\Delta\theta_i &= \sum_{i=1}^{7} \frac{\chi_{ij} r_j^1}{i}
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_{1j} &= 1 \\
\chi_{2j} &= 3a_j \\
\chi_{3j} &= 3(a_j^2 + b_j) \\
\chi_{4j} &= a_j(6b_j + a_j^2) \\
\chi_{5j} &= 3b_j i \\
\chi_{6j} &= 3a_j b_j^2 \\
\chi_{7j} &= b_j^3
\end{aligned}
$$

where all j subscripts denote the following:
$j = 1$ wet component
$j = 2$ dry component
Azimuth is not affected by refraction.

Optical Frequency

For optical frequencies, the wet term is not used. Therefore, the preceding formulas using only the dry term hold for Range, Range Rate, and Elevation Angle Corrections with optical frequencies.

Radio Frequency

For radio frequencies, it should be noted that μ (in microns) is much greater than 1, and therefore $\frac{1}{\mu^2}$ is considered to be zero.

Direction Cosines

The corrections Δl and Δm are derived from the elevation correction, $\Delta\theta$, for radio data

$$
\begin{aligned}
\Delta l &= \text{-}\sin A_z \sin(\theta)\Delta\theta \\
\Delta m &= \text{-}\cos A_z \sin(\theta)\Delta\theta
\end{aligned}
$$

where A_z is the azimuth angle computed from the initial estimate of the trajectory, and θ is the elevation angle.

X and Y Angles

For X and Y angles the corrections ΔX and ΔY are computed as follows:

$$\Delta X_a = -\frac{\sin(A_z)\Delta\theta}{(\sin^2(\theta)+\sin^2(A_z)\cos^2(\theta))}$$

$$\Delta Y_a = -\frac{\cos(A_z)\sin(\theta)\Delta\theta}{\sqrt{1-\cos^2(A_z)\cos^2(\theta)}}$$

Note that these are also derived from the elevation correction.

7.6 TRANET DOPPLER OBSERVATIONS

TRANET Doppler observations are received as a series of measured frequencies with an associated base frequency for each station pass. Using the following relationship, the GEODYN system converts these observations to range rate measurements:

$$\dot{\rho} = \frac{c(F_B - F_M)}{F_M}$$

where

F_M is the measured frequency,

F_B is the base frequency,

and

c is the velocity of light.

7.7 SATELLITE-SATELLITE TRACKING DATA PREPROCESSING

The preprocessing on the satellite-satellite tracking involves the determination of all the appropriate transit times. Because of the station- satellite and inter-satellite distances, this process must be performed iteratively. The required times are computed during the first iteration and are then stored for use ill subsequent iterations. The satellite-satellite tracking measurements are also corrected for tropospheric refraction. The corrections made here are identical to those which would be made on range and range rate measurements to the relay satellite only. Although it is theoretically possible for signals from the relay to low altitude satellite to pass through the atmosphere, such tracking would occur at reduced signal intensity and would be equivalent to the low elevation tracking of satellite from ground based stations. Such data is seldom used in orbit estimation. The standard procedure for transponder delay corrections on satellite-satellite tracking is to use block data constants for each satellite, with a satellite ID used to identify the appropriate block data entries. Since constants for the transponders to be used for satellite-satellite tracking are not presently available the block data entries must be modified appropriately when the data becomes available.

7.8 CORRECTIONS TO TIME MEASUREMENTS DUE TO THE EFFECTS OF GENERAL RELATIVITY

When the center of mass for orbit integration is the sun, the effects of general relativity, in particular the measurable bending of light when it passes near a very massive body, must be taken into account.

The equation for the time it takes light to travel from a satellite orbiting the sun at a point i, to an earth receiving station at a point j, taking into account the bending of light due to general relativity is given by:

$$t_j - t_i = \frac{r_{ij}}{c} + 2\frac{\mu_s}{c^3} ln \left[\frac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}} \right] \tag{7.8 -1}$$

If $r_i^s(t_i)$ and $r_j^s(t_j)$ are the heliocentric position vectors of the satellite at transmission time t_i and the earth receiver at reception time t_j, then

$$\begin{aligned} r_{ij} &= |r_j^s(t_j) - r_i^s(t_i)| \\ r_i &= |r_i^s(t_i)| \\ r_j &= |r_j^s(t_j)| \end{aligned}$$

Also

C = the speed of light

μ_s = the gravitational constant of the Sun.

The second term in (7.8-1) is the correction to the light time due to special relativity. General relativity considers time as one coordinate in the space-time coordinate systems of the satellite and the earth receiving stations. Thus an additional relativistic correction must be applied to the measured times.

For example, if we consider a range measurement for a signal transmitted at a coordinate time t_1 from the earth, received and retransmitted at the satellite at its coordinate time t_2, and finally received at an earth receiving station at its coordinate time t_3 , the total round trip time given by (1) is

$$t_3 - t_1 = \frac{r_{12} + r_{23}}{c} + 2\frac{\mu_s}{c^3} ln \left[\left[\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right] \left[\frac{r_2 + r_3 + r_{21}}{r_2 + r_3 - r_{23}} \right] \right] \tag{7.8 -2}$$

Since the measurements on the earth are made-using docks which keep A.1 atomic clock time, we define t_3^A as the A.l time at the receiving site at coordinate time t_3 , and t_1^A as the A.1 time at the transmitting site at its coordinate time t_1. The measured range value in this case is:

$$R_{meas} = \frac{C(t_3^A - t_1^A)}{2} \tag{7.8 -3}$$

and the calculated range value R_{calc}, with the corrections due to general relativity, is given by:

$$R_{calc} = R_{meas} - \frac{C(t_3^A - t_3 - t_1^A + t_1)}{2} - \frac{\mu_s}{C^2} ln \left[\left[\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right] \left[\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}} \right] \right] \tag{7.8 -4}$$

The correction to the range ΔR due to the difference between t^A and t is then given by:

$$\Delta R = \frac{-C}{2}(t_3^A - t_3 - t_1^A + t_1)$$

$$= \frac{C}{2}[1.658 \times 10^{-3}(\sin E_3 - \sin E_1) + 2.03^{-6} \cos u(\sin(UT_3 + \lambda) - \sin(UT_1 + \lambda))] \tag{7.8 -5}$$

Here

E_i = the eccentric anomaly of the heliocentric orbit of the earth-moon barycenter at time t_1

u = the distance of the atomic clock at the measurement site from the earth's spin axis in Km

UT_i =the Universal Time corresponding to t_i converted to radians

λ =the east longitude of the atomic clock

7.9 SCALING TO A NEW VALUE OF SPEED OF LIGHT

Each range, range rate or altimeter measurement is flagged with the speed of light used to convert the raw measurement to metric units. At present, GEODYN recognizes two values for the speed of light:
"Old" speed of light:

$$C_1 = 2.997925 \times 10^8 \frac{m}{sec}$$

"New" speed of light:

$$C_2 = 2.99792458 \times 10^8 \frac{m}{sec}$$

Each measurement is scaled to the internally used value which by default is C_2. The user may modify the built-in value via the VLIGHT option card. If C denotes the built-in value of the speed of light and C_i the value used for the measurement, the i^{th} measurement M_i is rescaled to:

$$M_i' = \frac{C}{C_i} M_i$$

7.10 RELATIVISTIC SPACECRAFT CLOCK DRIFT CORRECTIONS

The range correction due to the space-craft (S/C) relativistic clock drift only applies to Geodyn range and average range rate measurement types: 39, 40, 41, 42 and 55. See RELTMC card in volume 3 of the Geodyn documentation for an explanation on how to invoke this measurement correction.

The rate of a clock aboard a S/C is related to the rate of an earthbound station clock, including general relativity, by [7 -7] :

$$\tau_S = \tau + \frac{1}{c^2}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x_{STA}^2 + Y_{STA}^2) - \frac{3}{2}\frac{GM}{a}\right)\tau - \frac{2}{c^2}(\vec{x}_S \cdot \vec{c}_S) \qquad (7.10\ \text{-1})$$

where:

τ_s = S/C clock time

t = earth bound station clock time

C = speed of light

GM = GM of earth

r_{STA} = magnitude of position vector of earthbound station in geocentric coordinates.

Ω = rotation rate of the earth

x_{STA} = x component of an earthbound station in geocentric coordinates.

y_{STA} = y component of an earthbound station in geocentric coordinates.

a = semi-major axis of S/C orbit

$\vec{x}_s$ = position vector of S/C in geocentric coordinates.

$\vec{v}_s$ = velocity vector of S/C in geocentric coordinates.

For a one-way range measurement that is constructed from a signal that is emitted from the S/C and received at the station (Geodyn measurement types: 41 and 55) we have:

$$\tau_{(R)} - \tau_{s(e)} = \tau_R - \left[\tau_{(e)} + \frac{1}{c^2}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + Y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\tau_{(e)} - \frac{2}{c^2}(\vec{x}_S \cdot \vec{c}_S)\right] \tag{7.10 -2}$$

where:

$\tau_{(R)}$ = receive time from earthbound station clock

$\tau_{(e)}$ = emitted time from S/C earthbound station dock time

Multiplying by the speed of light and arranging terms, equation (7.10-2) becomes:

$$\rho = \rho_{calc} - \frac{1}{c}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\tau_{(e)} + \frac{2}{c}(\vec{x}_S \cdot \vec{v}_S) \tag{7.10 -3}$$

where:

ρ = measured range

ρ_{calc} = classically calculated range

Thus, the correction to a one-way range measurement, which is constructed from a signal that is emitted from a S/C and received at a station, is given by:

$$\rho_{\substack{corr \\ SAT \to STA}} = -\frac{1}{c}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\tau_{(e)} + \frac{2}{c}(\vec{x}_S \cdot \vec{v}_S) \tag{7.10 -4}$$

It should be noted that $\tau_{(e)}$ is the time that the signal was emitted from the time that the station and S/C clocks were synchronized. For the purposes of GEODYN, the synchronization time is the epoch time of the run. Thus,

$$\tau_{(e)} = \text{Modified julian day seconds OF (signal emission time) - (epoch time of run)} \tag{7.10 -5}$$

For GPS S/C the clocks were synchronized to eliminate the secular drift, thus, the one-way range correction for GPS S/C is given by:

$$\rho_{\substack{corr \\ SAT \to STA \\ GPS}} = \frac{2}{c}(\vec{x}_S \cdot \vec{v}_S) \tag{7.10 -6}$$

For a one-way measurement that is constructed from a signal which is emitted from the station and received at the S/C (Geodyn measurement type: 39) equation (7.10-2) becomes:

$$\tau_{S(R)} - \tau_{(e)} = \left[\tau_{(R)} + \frac{1}{c^2}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + Y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\tau_{(R)} - \frac{2}{c^2}(\vec{x}_S \cdot \vec{v}_S)\right] - \tau_{(e)} \quad (7.10\text{ -7})$$

Then the one-way range correction for a signal which is emitted from the station and received at the S/C is:

$$\rho_{\overset{corr}{STA\to SAT}} = \frac{1}{c}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\tau_{(R)} - \frac{2}{c}(\vec{x}_S \cdot \vec{v}_S) \quad (7.10\text{ -8})$$

$\tau_{(R)}$ = Modified julian day seconds OF (signal receive time) - (epoch time of run)

and for GPS:

$$\rho_{\underset{GPS}{\overset{corr}{STA\to SAT}}} = -\frac{2}{c}(\vec{x}_S \cdot \vec{v}_S) \quad (7.10\text{ -9})$$

Geodyn average range rates are constructed as follows:

$$\bar{\dot{\rho}} = \frac{\rho(t) - \rho(t+\Delta t)}{\Delta t} \quad (7.10\text{ -10})$$

If we include the range correction due to the relativistic S/C clock drift then (7.10-10) becomes:

$$\bar{\dot{\rho}} = \frac{\rho(t) + \rho_{corr}(t) - \rho(t+\Delta t) - \rho_{corr}(t+\Delta t)}{\Delta t} \quad (7.10\text{ -11})$$

Using equation (7.10-4) and (7.10-10) the expression for the the average range rate correction constructed from signals that emanate from the SIC and are received at an earthbound station (Geodyn measurement type: 41) is:

$$\begin{aligned}\bar{\dot{\rho}}_{\overset{corr}{SAT\to STA}} = &\left[\frac{1}{c}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x^2_{STA} + y^2_{STA}) - \frac{3}{2}\frac{GM}{a}\right)\right] \\ &+ \frac{\frac{2}{c}[(\vec{x}_S(t)\cdot\vec{v}_S(t)) - (\vec{x}_S(t+\Delta t)\cdot\vec{v}_S(t+\Delta t))]}{\Delta t}\end{aligned} \quad (7.10\text{ -12})$$

and for GPS:

$$\rho_{\substack{SAT \to STA \\ GPS}}^{corr} = \frac{\frac{2}{c}[(\vec{x}_S(t) \cdot \vec{v}_S(t)) - (\vec{x}_S(t+\Delta t) \cdot \vec{v}_S(t+\Delta t))]}{\Delta t} \tag{7.10 -13}$$

Derived in the same manner as above, the expression for the average range rate correction constructed from signals that eminate from an earthbound station and are received at the S/C (Geodyn measurement type: 40) is:

$$\begin{aligned} \bar{\dot{\rho}}_{STA \to SAT}^{corr} = & \left[-\frac{1}{c}\left(\frac{GM}{r_{STA}} + \frac{1}{2}\Omega^2(x_{STA}^2 + y_{STA}^2) - \frac{3}{2}\frac{GM}{a}\right)\right] \\ & - \frac{\frac{2}{c}[(\vec{x}_S(t) \cdot \vec{v}_S(t)) - (\vec{x}_S(t+\Delta t) \cdot \vec{v}_S(t+\Delta t))]}{\Delta t} \end{aligned} \tag{7.10 -14}$$

and for GPS:

$$\rho_{\substack{STA \to SAT \\ GPS}}^{corr} = \frac{-\frac{2}{c}[(\vec{x}_S(t) \cdot \vec{v}_S(t)) - (\vec{x}_S(t+\Delta t) \cdot \vec{v}_S(t+\Delta t))]}{\Delta t} \tag{7.10 -15}$$

8 FORCE MODEL AND VARIATIONAL EQUATIONS

A fundamental part of the GEODYN system requires computing positions and velocities of the spacecraft at each observation time. The dynamics of the situation are expressed by the equations of motion, whlch provide a relationship between the orbital elements at any given instant and the initial conditions of epoch. There is an additional requirement for variational partials, which are the partial derivatives of the instantaneous orbital elements with respect to the parameters at epoch. These partials are generated using the variational equations, which are analogous to the equations of motion.

8.1 EQUATIONS OF MOTION

In a geocentric inertial rectangular coordinate system, the equations of motion for a spacecraft are of the form

$$\ddot{\bar{r}} = -\frac{\mu\vec{r}}{r^3} + \bar{A} \tag{8.1 -1}$$

where

$\bar{r}$ is the position vector of the satellite

μ is GM, where G is the gravitational constant and M is the mass of the Earth,

$\bar{A}$ is the acceleration caused by the asphericity of the Earth, extraterrestrial gravitational forces, atmospheric drag, and solar radiation

This provides a system of second order differential equations which, given the epoch position and velocity components, may be integrated to obtain the position and velocity at any other time. This direct integration of these accelerations in Cartesian coordinates is known as Cowell's method and is the technique used in GEODYN's orbit generator. This method was selected for its simplicity and its capacity for easily incorporating additional perturbative forces.
There is an alternative way of expressing the above equations of motion:

$$\ddot{\bar{r}} = \nabla U + \bar{A}_D + \bar{A}_R \qquad (8.1\text{ -}2)$$

where

U is the potential field due to gravity,

$\bar{A_D}$ contains the accelerations due to drag, and

$\bar{A_R}$ contains the accelerations due to solar the accelerations due to solar radiation pressure.

This is, of course, just a regrouping of terms coupled with a recognition of the existence of a potential field. This is the form used in GEODYN. The inertial coordinate system in which these equations of motion are integrated in GEODYN is that system corresponding to the true of date system of $0.^h0$ of the reference day. The complete definitions for these coordinate systems (and the Earth-fixed system) are presented in Section 3.0. The evaluation of the accelerations for $\ddot{\bar{r}}$ is performed in the true of date system. Thus there is a requirement that the inertial position and velocity output from the integrator be transformed to the true of date system for the evaluation of the accelerations, and a requirement to transform the computed accelerations from the true of date system to the inertial system.

8.2 THE VARIATIONAL EQUATIONS

The variational equations have the same relationship to the variational partials as the satellite position vector does to the equations of motion.

The variational partials are defined as $\frac{\partial \bar{x}_t}{\partial \beta}$ where $\bar{x}_t$ spans the true of date position and velocity of the satellite at a given time; i.e.,

$$\bar{x}_t = (x, y, z, \dot{x}, \dot{y}, \dot{z})$$

and $\bar{\beta}$ spans the epoch parameters; i.e.,

x_0, y_0, z_0 the satellite position vector at epoch

$\dot{x}_0, \dot{y}_0, \dot{z}_0$ the satellite velocity vector at epoch

C_D the satellite drag factor

$\dot{C}_D$ the time rate of change of the drag factor

C_R the satellite emissivity factor

C_{nm}, S_{nm} gravitational harmonic coefficients for each n, m pair being determined

X surface density coefficients

Let us first realize that the variational partials may be partitioned according to the satellite position and velocity vectors at the given time. Thus the required partials are

$$\frac{\partial \bar{r}}{\partial \bar{\beta}}, \frac{\partial \dot{\bar{r}}}{\partial \bar{\beta}} \tag{8.2 -1}$$

where

$\bar{r}$ is the satellite position vector (x,y, z) in the true of date system, and

$\dot{\bar{r}}$ is the satellite velocity vector (x, y, z) in the same system

The first of these, $\frac{\partial \bar{r}}{\partial \bar{\beta}}$, can be obtained by the double integration of

$$\frac{d^2}{dt^2}\left[\frac{\partial \bar{r}}{\partial \bar{\beta}}\right] \tag{8.2 -2}$$

or rather, since the order of differentiation may be exchanged,

$$\frac{\partial \ddot{\bar{r}}}{\partial \bar{\beta}} \tag{8.2 -3}$$

Note that the second set of partials, $\frac{\partial \dot{\bar{r}}}{\partial \bar{\beta}}$, may be obtained by a first order integration of $\frac{\partial \ddot{\bar{r}}}{\partial \bar{\beta}}$. Hence we recognize that the quantity to be integrated is $\frac{\partial \ddot{\bar{r}}}{\partial \bar{\beta}}$. Using the second form given for the equations of motion in the previous subsection, the variational equations are given by

$$\frac{\partial \ddot{\bar{r}}}{\partial \bar{\beta}} = \frac{\partial}{\partial \bar{\beta}}(\nabla U + \bar{A}_R + \bar{A}_D) \tag{8.2 -4}$$

where

U is the potential field due to gravitational effects

$\bar{A}_R$ is the acceleration due to radiation pressure

$\bar{A}_D$ is the acceleration due to drag

The similarity to the equations of motion is now obvious.
At this point we must consider a few items:

- The partial field is a function only of position. Thus we have

$$\frac{\partial}{\partial \bar{\beta}} \left[\frac{\partial U}{\partial r_i} \right] = \sum_{m=1}^{3} \left[\frac{\partial^2 U}{\partial r_i \partial r_m} \right] \frac{\partial r_m}{\partial \bar{\beta}} \tag{8.2 -5}$$

- The partials of solar radiation pressure with respect to the geopotential coefficients, the drag coefficient, and the satellite velocity are zero, and the partials, with respect to satellite position, are negligible.

- Drag is function of Position, velocity, and the drag coefficients. The partials, with respect to the geopotential coefficients and satellite emissivity, are zero, but we have

$$\frac{\partial \bar{A}_D}{\partial \bar{\beta}} = \frac{\partial A_D}{\partial \bar{x}_t} \frac{\partial \bar{x}_t}{\partial \bar{\beta}} + \frac{\partial A_D}{\partial \bar{C}_D} \frac{\partial \bar{C}_D}{\partial \bar{\beta}} + \frac{\partial A_D}{\partial \dot{\bar{C}}_D} \frac{\partial \dot{\bar{C}}_D}{\partial \bar{\beta}} \tag{8.2 -6}$$

Let us write our variational equations in matrix notation. We define

n to be the number of epoch parameters in $\bar{\beta}$

F is a 3 x 11 matrix whose j^{th} column vectors are $\frac{\partial \ddot{r}}{\partial \beta_j}$

U_{2c} is a 3 x 6 matrix whose last 3 columns are zero and whose first 3 columns are such that the i, j^{th} element is given by $\frac{\partial^2 U}{\partial r_i \partial r_j}$

D_r is a 3 x 6 matrix whose i_{th} column is defined by $\frac{\partial \bar{x}_i}{\partial x_{ij}}$

X_m s a 6 x n matrix whose i^{th} row is given by $\frac{\partial \bar{x}_i}{\partial \beta_i}$. Note that X_m contains the variational partials

f is a 3 x n matrix whose first six columns are zero and whose last $n-6$ columns are such that the i, j^{th} element is given by $\frac{\partial}{\partial \beta_j} (\nabla U + \bar{A}_D + \bar{A}_R) i$. Note that the first six columns correspond to the first six elements of $\bar{\beta}$ which are the epoch and velocity. (This matrix contains the direct partials of $\bar{x}_t$, with respect to $\bar{\beta}$).

We may now write

$$F = [U_{2c} + D_r]X_m + f \qquad (8.2\ \text{-7})$$

This is a matrix form of the variational equations.
Note that U_2c, D_r and f are evaluated at the current time, whereas X_m is the output of the integration. lnitially, the first six columns of X_m plus the six rows form an identity matrix; the rest of the matrix is zero (for $i = j, X_mi, j = 1$; for $i = j, X_mi, j = 0$).

Because each force enters into the variational equations in a manner which depends directly on its model, the specific contribution of each force is discussed in the section with the force model. We shall, however, note a few clerical details here.

The above equation is also applied in subroutine PREDCT to "chain the partials back to epoch," that is, to relate the partials at the time of each set of measurements back to epoch.

The matrix for $\frac{\partial \bar{x}_t}{\partial \bar{\beta}}$, X_m above, is initialized in subroutine ORBIT.

The matrices U_{2c} and f will be referred to in each of the following subsections as though the particular for e being discussed had the only contribution.

8.3 THE EARTH'S POTENTIAL

In GEODYN the Earth's potential is described by the combination of a spherical harmonic expansion and a surface density layer. Generally, however, the spherical harmonic expansion is used exclusively at no surface density terms are included.

8.3.1 Spherical Harmonic Expansion

The Earth's potential is most conveniently expressed in a spherical coordinate system as is shown in Figure 8.3-1. By inspection:

- ϕ', the geocentric latitude, is the angle measured from $\bar{OA}$,the projection of $\bar{OP}$ in X - Y plane, to the vector $\bar{OP}$.
- λ, the east longitude, is the angle measured from the positive direction of the X axis to $\bar{OA}$
- r, is the magnitude of the vector $\bar{OP}$.

Let us consider the point P to be the satellite position. Thus, $\bar{OP}$ is the geocentric Earth-fixed satellite vector corresponding to $\bar{r}$ the true of date satellite vector, whose components are (x, y, z). The relationship between the spherical coordinates (Earth-fixed) and the satellite position coordinates (true of date) is then given by

$$r = \sqrt{x^2 + y^2 + z^2} \tag{8.3 -1}$$

$$\phi' = \sin^{-1}[\frac{z}{r}] \tag{8.3 -2}$$

$$\lambda = \tan^{-1}[\frac{y}{x}] - \theta_g \tag{8.3 -3}$$

where θ_g is the rotation angle between the true of date system and the Earth- fixed system (see Section 3.4).

The Earth's gravity field is represented by the normal potential of an ellipsoid of revolution and small irregular variations, expressed by a sum of spherical harmonics. This formulation, used in the GEODYN systern, is

$$U = \frac{GM}{r}\left[1 + \sum_{n=2}^{nmax}\sum_{m=0}^{n}\left[\frac{a_e}{r}\right]^n P_m^n(\sin\phi')[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right] \tag{8.3 -4}$$

where

G is the universal gravitational constant,

M is the mass of the earth,

r is the geocentric satellite distance,

n max is the upper limit for the summation (highest degree),

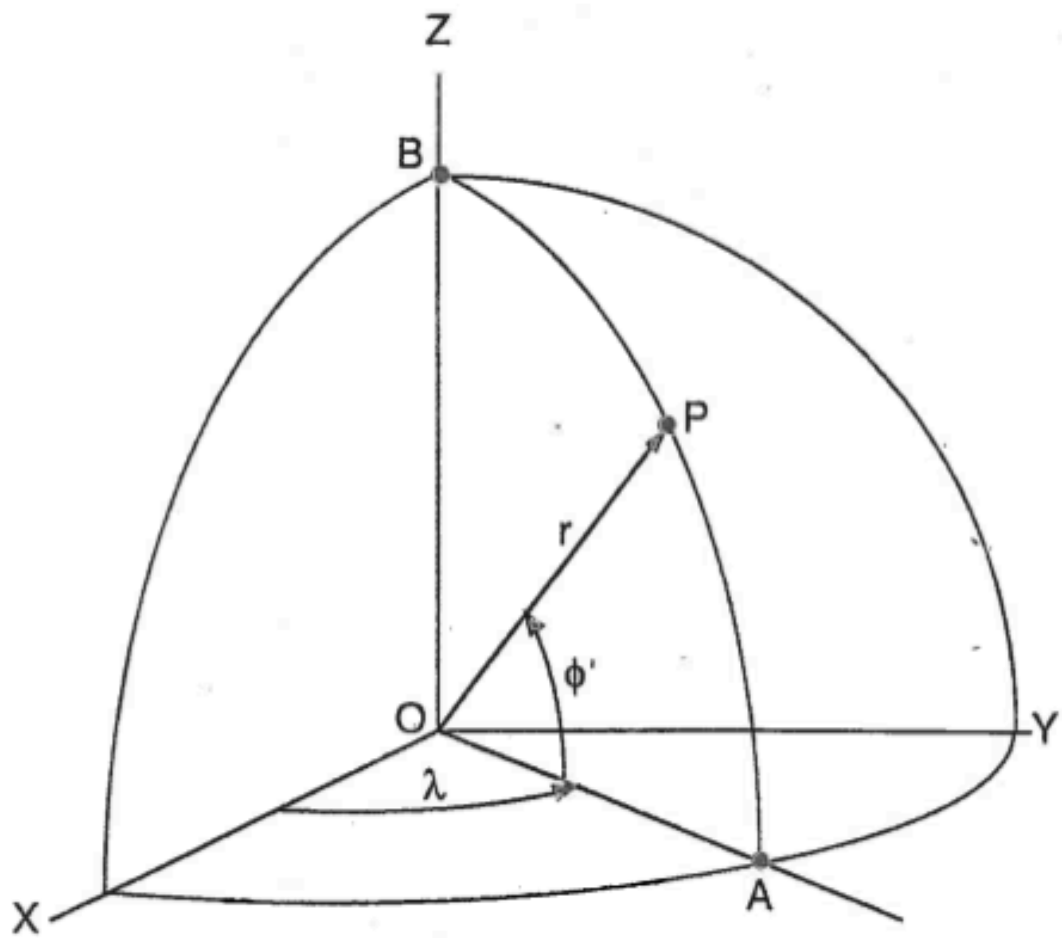
a_e is the Earth's mean equatorial radius,

ϕ' is the satellite geocentric latitude,

λ is the satellite east longitude,

$P_m^n(\sin\phi')$ indicate the associated Legendre functions, and

C_{nm} and S_{nm} are the denormalized gravitational coefficients.


Z
B
P
r
φ'
O
Y
λ
X
A

The relationships between the normalized coefficients $(\bar{C}_{nm}, \bar{S}_{nm})$ and the denormalized coefficients are as follows:

$$C_{nm} = \left[\frac{(n-m)!(2n+1)(2-\delta_{om})}{(n+m)!}\right]^{\frac{1}{2}} \bar{C}_{nm} \qquad (8.3\text{ -5})$$

where

δ_{om} is the Kronecker delta,

$\delta = 1$ for $m = 0$ and $\delta_{om} = 0$ for $m \neq 0$.

A similar expression is valid for the relationship between $\bar{S}_{nm}$ and S_{nm}.

The gravitational accelerations in true of date coordinates $(\ddot{x}, \ddot{y}, \ddot{z})$ are computed from the geopotential, $U(r, \phi', \lambda)$ by the chain rule; e.g.,

$$\ddot{x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \phi'}\frac{\partial \phi'}{\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x} \qquad (8.3\text{ -6})$$

The accelerations $\ddot{y}$ and $\ddot{z}$ are determined likewise. The partial derivatives of U with respect to r, ϕ', and λ, are given by

$$\frac{\partial U}{\partial r} = \frac{GM}{r}\left[1 + \sum_{n=2}^{nmax}\left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \times (n+1)P_m^n(\sin\phi')\right] \qquad (8.3\text{ -7})$$

$$\frac{\partial U}{\partial \lambda} = \frac{GM}{r}\sum_{n=2}^{nmax}\left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda) \times m P_m^n(\sin\phi') \qquad (8.3\text{ -8})$$

$$\frac{\partial U}{\partial \phi'} = \frac{GM}{r}\sum_{n=2}^{nmax}\left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \times \left[P_m^n(\sin\phi') - m\tan\phi' P_m^n(\sin\phi')\right] \qquad (8.3\text{ -9})$$

The partial derivatives of r, ϕ', and λ with respect to the true of date satellite position components are

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r} \tag{8.3 -10}$$

$$\frac{\partial \phi'}{\partial r_i} = -\frac{1}{sqrt x^2 + y^2}\left[\frac{z r_i}{r^2} - \frac{\partial z}{\partial r_i}\right] \tag{8.3 -11}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{x^2 + y^2}\left[x\frac{\partial y}{\partial r_i} - y\frac{\partial x}{\partial r_i}\right] \tag{8.3 -12}$$

(The Legendre functions are computed via recursion formulae:
Zonals: m= 0

$$P_n^0(\sin\phi') = \frac{1}{n}[(2n-1)\sin\phi' P_{n-1}^0(\sin\phi') - (n-1)P_{n-2}^0(\sin\phi')] \tag{8.3 -13}$$

$$P_1^0(\sin\phi') = \sin\phi' \tag{8.3 -14}$$

Tesserals: m $\neq$ 0, m *leq* n

$$P_n^m(\sin\phi') = P_{n-2}^m(\sin\phi') + (2n-1)\cos\phi' P_{n-1}^{m-1}(\sin\phi')] \tag{8.3 -15}$$

$$P_1^1(\sin\phi') = \cos\phi' \tag{8.3 -16}$$

Sectorials m = n

$$P_n^m = (2n-1)\cos\phi' P_{n-1}^{m-1}(\sin\phi')] \tag{8.3 -17}$$

The derivative relationship is given by:

$$\frac{d}{d\phi'}[P_n^m(\sin\phi')] = P_n^{m+1}(\sin\phi') - m\tan\phi' P_n^m(\sin\phi') \tag{8.3 -18}$$

It should also be noted that multiple angle formulas are used for evaluating the sine and cosine of $m\lambda$.

Arrays containing certain intermediate data are used in the computations for the variational equations. These contain the values for:

$$\frac{GM}{r}\left[\frac{a_e}{r}\right]^n \tag{8.3 -19}$$

$$P_n^m(\sin\phi')$$

$$\sin m\lambda$$

$$\cos m\lambda$$

$$m\tan\phi'$$

for each m and/or n.

The variation equations require the computation of the matrix U_{2c} ,whose elements are given by

$$[U_{2c}]_{i,j} = \frac{\partial^2 U}{\partial r_i \partial r_j} \tag{8.3 -20}$$

where

r_i =the true of date satellite position

U =is the geopotential.

Because the Earth's field is in terms of $r, \sin\phi'$ and λ, we write

$$U_{2c} = C_1^T U_2 C_1 + \sum_{k=1}^{3} \frac{\partial U}{\partial e_k} C_{2k} \tag{8.3 -21}$$

where

e_k ranges over the elements r

U_2 is the matrix whose i, j^{th} element is given by $\partial^2 U \partial e_i \partial e_j$

C_1 is the matrix whose i, j^{th} element is given by $\frac{\partial e_i}{\partial r_j}$

C_{2k} is the matrix whose i, j^{th} element is given by $\frac{\partial^2 e_k}{\partial r_i \partial r_j}$

We compute the second partial derivatives of the potential U with respect to r, ϕ' and λ:

$$\frac{\partial^2 U}{\partial r^2} = \frac{2GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{nmax} (n+1)(n+2) \left[\frac{a_e}{r}\right]^n \times \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin\phi') \tag{8.3 -22}$$

$$\frac{\partial^2 U}{\partial r \partial \phi'} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin\phi') \tag{8.3 -23}$$

$$\frac{\partial^2 U}{\partial r \partial \lambda} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m(-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin\phi') \tag{8.3 -24}$$

$$\frac{\partial^2 U}{\partial \phi'^2} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \frac{\partial^2}{\partial \phi'^2} [P_n^m(\sin\phi')] \tag{8.3 -25}$$

$$\frac{\partial^2 U}{\partial \phi' \partial \lambda} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m(-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin\phi') \tag{8.3 -26}$$

$$\frac{\partial^2 U}{\partial \lambda^2} = -\frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m^2(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin\phi') \tag{8.3 -27}$$

where

$$\frac{\partial}{\partial \phi'}[P_n^m(\sin\phi')] = P_n^{m+1}(\sin\phi') - m \tan\phi' P_n^m(\sin\phi') \tag{8.3 -28}$$

$$\frac{\partial^2}{\partial \phi'}[P_n^m(\sin\phi')] = P_n^{m+2}(\sin\phi') - (m+1)\tan\phi' P_n^{m+1}(\sin\phi') - m \tan\phi'[P_n^{m+1}(\sin\phi') - m\tan\phi' P_n^m(\sin\phi')] - m \sec^2\phi' P_n^m(\sin\phi') \tag{8.3 -29}$$

The elements of U_2 have almost been computed. What remains is to transform from (r, ϕ', λ) to $(r, \sin\phi', \lambda)$. This affects only the partials involving ϕ':

$$\frac{\partial U}{\partial \sin\phi'} = \frac{\partial U}{\partial \phi'}\frac{\partial \phi'}{\partial \sin\phi'} \tag{8.3 -30}$$

$$\frac{\partial^2 U}{\partial \sin\phi'^2} = \frac{\partial \phi'}{\partial \sin\phi'}\left[\frac{\partial^2 U}{\partial \phi'^2}\right]\frac{\partial \phi'}{\partial \sin\phi'} + \frac{\partial U}{\partial \phi'}\frac{\partial^2 \phi'}{\partial \sin\phi'^2} \tag{8.3 -31}$$

where

$$\frac{\partial \phi'}{\partial \sin\phi} = \sec\phi' \tag{8.3 -32}$$

$$\frac{\partial^2 \partial\phi'}{\partial \sin\phi'^2} = \sin\phi' \sec^3\phi' \tag{8.3 -33}$$

For the C_1 and C_{2k} matrices, the partials of $r, \sin\phi'$, and λ are obtained from the usual formulas:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{8.3 -34}$$

$$\sin\phi' = \frac{z}{r} \tag{8.3 -35}$$

$$\lambda = \tan^{-1}\left[\frac{y}{x}\right] - \theta_g \tag{8.3 -36}$$

We have for C_1:

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r} \tag{8.3 -37}$$

$$\frac{\partial \sin\phi'}{\partial r_i} = -\frac{z r_i}{r^3} + \frac{1}{r}\frac{\partial x}{\partial r_i} \tag{8.3 -38}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{x^2 + y^2}\left[x\frac{\partial y}{\partial r_i} - y\frac{\partial x}{\partial r_i}\right] \tag{8.3 -39}$$

The C_{2k} are symmetric. The necessary elements are given by

$$\frac{\partial^2 r}{\partial r_i \partial r_j} = \frac{r_i r_j}{r^3} + \frac{1}{r}\frac{\partial r_i}{\partial r_j} \tag{8.3 -40}$$

$$\frac{\partial^2 \sin\phi'}{\partial r_i \partial r_j} = \frac{3 z r_i r_j}{r^5} - \frac{1}{r^3}\left[r_j\frac{\partial z}{\partial r_i} + r_i\frac{\partial z}{\partial r_j} + z\frac{\partial r_i}{\partial r_j}\right] \tag{8.3 -41}$$

$$\frac{\partial^2 \lambda}{\partial r_i \partial r_j} = \frac{-2 r_j}{(x^2 + y^2)^2}\left[x\frac{\partial y}{\partial r_i} - y\frac{\partial x}{\partial r_i}\right] + \frac{1}{(x^2 + y^2)}\left[\frac{\partial x}{\partial r_j}\frac{\partial y}{\partial r_j} - \frac{\partial y}{\partial r_j}\frac{\partial x}{\partial r_j}\right] \tag{8.3 -42}$$

If gravitational constants, C_{nm} or S_{nm} are being estimated, we require their partials in the f matrix for the variational equations computations. These partials are

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial r}\right] = (n+1)\frac{GM}{r^2}\left[\frac{a_e}{r}\right]^n \cos(m\lambda) P_n^m(\sin\phi') \tag{8.3 -43}$$

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \lambda}\right] = m\frac{GM}{r}\left[\frac{a_e}{r}\right]^n \sin(m\lambda) P_n^m(\sin\phi') \tag{8.3 -44}$$

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \phi'}\right] = \frac{-GM}{r}\left[\frac{a_e}{r}\right]^n \cos(m\lambda)\left[P_n^{m+1}(\sin\phi') - m\tan\phi' P_n^m(\sin\phi')\right] \tag{8.3 -45}$$

The partials for S_{nm} are identical with $\cos(m\lambda)$ replaced by $\sin(m\lambda)$ and with $\sin(m\lambda)$ replaced by $-\cos(m\lambda)$. These partials are converted to inertial true of date coordinates using the chain rule; e.g.,

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial x}\right] = \frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial r}\right]\frac{\partial r}{\partial x} + \frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \lambda}\right]\frac{\partial \lambda}{\partial x} + \frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \phi'}\right]\frac{\partial \phi'}{\partial x} \tag{8.3 -46}$$

8.3.1.1 Lumped Sum Gravity Model Difference Partial For certain error analysis studies it is more important to know the effect of an error resulting from different gravity models than it is to know the effect due to an error in a particular geopotential coefficient. GEODYN will generate a lumped sum gravity model difference partial in the following manner. The individual geopotential coefficients are differenced to form

$$\begin{aligned} \Delta C_N^M &= C_{Nmodel1}^M - C_{Nmodel2}^M \left[\frac{a_e(model1)}{a_e(model2)}\right]^3 \\ \Delta S_N^M &= S_{Nmodel1}^M - S_{Nmodel2}^M \left[\frac{a_e(model1)}{a_e(model2)}\right]^3 \end{aligned} \tag{8.3 -47}$$

where the ratio of $a_e(model1)$ to $a_e(model2)$ is used to scale the second model's coefficients to the same reference system as the first model's coefficients. The partial of the satellite's acceleration with respect to the lumped sum gravity model difference is given by

$$\frac{\partial \ddot{X}}{\partial LSGMD} = \sum_{N=1}^{NMAX}\sum_{M=0}^{N}\frac{\partial \ddot{X}}{\partial C_N^M}\Delta C_N^M + \sum_{N=1}^{NMAX}\sum_{M=0}^{N}\frac{\partial \ddot{X}}{\partial S_N^M}\cdot \Delta S_N^M$$

The summation for the lumped sum partial does not include geopotential coefficients that arc estimated in the data reduction run. The partial of the satellite's position with respect to the lumped sum gravity m del difference is given by

$$\frac{\partial \ddot{X}}{\partial LSGMD} = \int\int \frac{\partial \ddot{X}}{\partial LSGMD} dtdt$$

8.3.2 Temporal Variations

To account for temporal variations in the gravity field the GEODYN model has been enhanced to allow linear rates and/or periodic terms to be specified for each of the geopotential coefficients. The formulation for the time dependent gravity implementation is as follows:

$$\bar{C}_{N,M} = \bar{C}_{N,M} + \dot{C}_{N,M}(t - t_0) + A_{C_{N,M}} \cos(\Omega_{N,M}(t - t_0)) + B_{C_{N,M}} \sin(\Omega_{N,M}(t - t_0))$$
$$\bar{S}_{N,M} = \bar{S}_{N,M} + \dot{S}_{N,M}(t - t_0) + A_{S_{N,M}} \cos(\Omega_{N,M}(t - t_0)) + B_{S_{N,M}} \sin(\Omega_{N,M}(t - t_0))$$

$\bar{C}_{N,M}, \bar{S}_{N,M}$ Normalized geopotential C and S coefficients.

$\dot{C}, \dot{S}$ Linear rates for the geopotential coefficients.

t_0 Start time for application of time dependent gravity

t Current time. Time at which time dependent gravity will be computed.

$A_{C_{N,M}}, A_{S_{N,M}}$ Amplitude of the cosine term for the $C_{N,M}$ or $S_{N,M}$ coefficient for application of a periodic gravity effect.

$B_{C_{N,M}}, B_{S_{N,M}}$ Amplitude of the sine term for the $C_{N,M}$ or $S_{N,M}$ coefficient for application of a periodic gravity effect.

8.4 SOLAR, LUNAR, AND PLANETARY GRAVITATIONAL PERTURBATIONS

The perturbations caused by a third body on a satellite orbit are treated by defining a function, R_d, which is the third body disturbing potential. This potential takes on the following form:

$$R_d = \frac{Gm_d}{r_d}\left[\left[1 - \frac{2r}{r_d}S + \frac{r^2}{r_d^2}\right]^{-\frac{1}{2}} - \frac{r}{r_d}S\right] \quad (8.4\text{ -1})$$

where

m_d is the mass of the disturbing body.

$\bar{r}_d$ is the geocentric true of date position vector to the disturbing body.

S is equal to the cosine of the enclosed angle between $\bar{r}$ and $\bar{r_d}$.

$\bar{r}$ is the geocentric true of date position vector of the satellite.

G is the universal gravitational constant.

The third body perturbations considered in GEODYN are for the Sun, the Moon and the planets.

$$\bar{a_d} = -Gm_d\left[\frac{\bar{d}}{D} + \left[\frac{\bar{r}}{r_d}\right]\right] \tag{8.4 -2}$$

where

$$\bar{d} = \bar{r} - \bar{r}_d$$

$$D_d = [r_d^2 - 2rr_dS + r^2]^{\frac{3}{2}}$$

These latter quantities, $\bar{d}$ and D_d as well as $D_d^{\frac{2}{3}}$ are used for the variational equation calculations. The j^{th} elements of the matrix U_{2C} are given by

$$\frac{\partial^2 R_d}{\partial r_i \partial r_j} = -\frac{GMm_d}{D_d}\left[\frac{\partial r_i}{\partial r_j} + \frac{3d_i d_j}{D_d^{\frac{2}{3}}}\right] \tag{8.4 -3}$$

This matrix is a fundamental part of the variational equations.

8.5 SOLAR RADIATION PRESSURE

The force due to solar radiation can have a significant effect on the orbits of satellites 'with a large area to mass ratio. GEODYN calculates A_R using average values for the cross-sectional area, A_s ,and the reflectivity coefficient C_R.
The accelerations due to solar radiation pressure are formulated as

$$\bar{A}_R = -vC_R\frac{A_S}{m_s}P_s\hat{r}_s \tag{8.5 -1}$$

where

v is the eclipse factor, such that

$v = 0$ when the satellite is in the Earth's shadow

$v = 1$ when the satellite is illuminated by the Sun

C_R is a factor depending on the reflective characteristics of the satellite,

A_s is the cross sectional area of the satellite,

m_s is the mass of the satellite,

P_s is the solar radiation pressure in the vicinity of the Earth, and

$\hat{r}_s$ is the (geocentric) true of date unit vector pointing to the Sun.

The unit vector $\hat{r}_s$ is determined as part of the luni-solar-planetary ephemeris computations.

The eclipse factor, v, is determined as follows: Compute

$$D = r \cdot \hat{r}_s \qquad (8.5\text{ -2})$$

where $\bar{r}$ is the true of date position vector of the satellite. If D is positive, the satellite is always in sunlight. If D is negative, compute the vector $\bar{P}_R$.

$$\bar{P}_R = \bar{r} - D\hat{r}_s \qquad (8.5\text{ -3})$$

This vector is perpendicular to r_s. If its magnitude is less than an Earth radius, or rather if

$$\bar{P}_R \cdot \bar{P}_R < a_e^2 \qquad (8.5\text{ -4})$$

the satellite is in shadow.

The satellite is assumed to be specularly reflecting with reflectivity ρ_s; thus

$$C_R = 1 + \rho_s \qquad (8.5\text{ -5})$$

When a radiation pressure coefficient is being determined; i.e., C_R, the partials for the f matrix in the variational equations computation must be computed. The i^{th} element of this column matrix is given by

$$f_i = -v \frac{A_s}{m_s} P_s r_{s_i} \qquad (8.5\text{ -6})$$

8.5.1 Radiation Pressure Models for GPS Satellites

Precise estimation and prediction of the positions of GPS satellites requires an accurate model of the solar pressure forces acting on the spacecraft. For Block I (earlier satellites) and Block II (new GPS satellites), the models used are ROCK4 and ROCK42, respectively.

They calculate the direct effect of the reflected sunlight from the spacecraft body and solar panels. They do not include the effect of absorbed and radiated heat, and are supplemented in practice by empirical estimation of a scaling factor in the radial direction, and of another quantity called Y -axis bias.

The Rockwell models are expressed in a spacecraft coordinate system. The + Z direction is toward the Earth, and therefore along the spacecraft antenna. The + X direction is toward the semi-circle that sees the sun, and + Y completes a right-banded system, and points along one of the solar panel center beams. There are four contributions considered to the solar pressure force.

1. All radiation is stopped by an opaque surface, therefore all the momentum of the incoming quanta is transferred to the spacecraft.
2. Part of the incoming radiation is reflected; In both ROCK4 and ROCK42, the recoil of reflected light is separated into two components, specular and diffuse; and the force normal to the reflecting surface is only 2/3 for the diffuse component what it is for the specular.
3. Part of the radiation absorbed by the solar panels is converted to electricity. Most of this energy will be radiated from the spacecraft body in the form of heat. It is not modeled in ROCK4, but a rough-and-ready thermal calculation is supplied with ROCK42.
4. Much of the incoming radiation is absorbed as heat and reradiated back into space by the absorbing surface. Both empirical models include this.

The X and Z force components (Y being accounted in the Y-bias effect) for the two models are the following [8 – 35]:

ROCK4:

$$X = -4.55\sin(A) + 0.08\sin(2A + 0.9) - 0.06\cos(4A + 0.08) + 0.08$$
$$Z = -4.54\cos(A) + 0.20\sin(2A - 0.3) - 0.03\sin(4A)$$

ROCK42:

$$X = -8.96\sin(A) + 0.16\sin(3A) + 0.10\sin(5A) - 0.07\sin(A)$$
$$Z = -8.43\cos(A)$$

where:

A is the angle between the Sun and the + Z direction that is the direction of the antenna pointing toward the Earth (solar angle).

The above models are used as default models in GEODYN II for radiation pressure on Block 1 and Block 2 GPS satellites. An earlier version yields a slightly different evaluation of reflectance and specularity coefficients. This version is also available in GEODYN II (See Geodyn II documentation, vol. 3, ROCK4 input card). The formulation for the earlier model is the following:

ROCK4:

$$X = -4.5\sin(A) + 0.08\sin(2A + 0.9) - 0.06\cos(4A + 0.08) + 0.08$$
$$Z = . - 4.5\cos(A) + 0.2\sin(2A - 0.3) - 0.03\sin(4A)$$

ROCK42:

$$X = -8.5\sin(A) + 0.018\sin(2A + 2) - 0.73\sin(4A + 1.3) + 0.07$$
$$Z = -8.2\cos(A) + 0.154\sin(2A - 0.3) - 0.052\sin(4A + 1.3) + 0.01$$

The force is translated into acceleration when divided by the spacecraft mass. In GEODYN II when GPS satellites are involved, the cannonball solar radiation model is completely ignored but the SOLRAD input card can still be used to introduce an adjustable solar radiation coefficient. The solar radiation coefficient simply multiplies the acceleration.

8.6 ATMOSPHERIC DRAG

A satellite moving through an atmosphere experiences a drag force. The acceleration due to this force is given by

$$\bar{A}_D = -\frac{1}{2} C_D \frac{A_s}{m_s} \rho_D v_r \bar{r}_r \qquad (8.6\text{ -1})$$

where

C_D is the satellite drag coefficient

A_s is the cross-sectional area of the satellite

m_s is the mass of the satellite

ρ_D is the-density of the atmosphere at the satellite position

$\bar{v}_r$ is the velocity vector of the satellite relative to the atmosphere, and

v_r is the magnitude of the velocity vector, $\bar{v}_r$

C_D is treated as a constant in GEODYN. The factor C_D varies slightly with satellite shape and atmospheric composition. However, for any geodetically useful satellite, it may be treated as a satellite dependent constant.

The relative velocity vector, $\bar{v}_r$, is computed assuming that the atmosphere rotates with the Earth. The true of date components of this vector are then

$$\dot{r}_r = \dot{x} + \dot{\theta}_g y \tag{8.6 -2}$$

$$\dot{y}_r = \dot{y} - \dot{\theta}_g x \tag{8.6 -3}$$

$$\dot{z}_r = \dot{z} \tag{8.6 -4}$$

as is indicated from Section 3.4, the subsection on transformation between Earth-fixed and true of date systems. The quantities $\dot{x}$, $\dot{y}$, and $\dot{z}$ are of course the components of $\dot{\bar{r}}$, the satellite velocity vector in true of date coordinates.

The cross-sectional area of the satellite, A_s is treated as a constant. For most work with geodetic satellites the use of a mean. cross-sectional area does not lead to significant errors.

The direct partials for the f matrix of the variational equations when the drag coefficient C_D is being determined given by

$$f = -\frac{1}{2}\frac{A_s}{m_s}\rho_D v_r \bar{v}_r \tag{8.6 -5}$$

When drag is present in an orbit determination run, the D_r matrix in the variational equations must also be computed. This matrix contains the partial derivatives of the drag acceleration with respect to the Cartesian orbital elements. We have

$$D_r = -\frac{1}{2}C_D\frac{A_s}{m_s}\left[\rho_D v_r \frac{\partial \bar{v}_r}{\partial \bar{x}_t} + \rho_D \frac{\partial v_r}{\partial \bar{x}_t}\bar{v}_r + \frac{\partial \rho_D}{\partial \bar{x}_t} v_r \bar{v}_r\right] \tag{8.6 -6}$$

where $\bar{x}_t$ is $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ i.e., $\bar{x}_t$, spans $\bar{r}$ and $\dot{\bar{r}}$.

$$\frac{\partial \bar{v}_r}{\partial \bar{x}_t} = \begin{bmatrix} 0 & -\dot{\theta}_g & 0 \\ \dot{\theta}_g & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.6 -7}$$

$$\frac{\partial \bar{v}_r}{\partial \bar{x}_t}\bar{v}_r = \frac{1}{v_r}\begin{bmatrix} -\dot{y}_r\dot{\theta}_g\dot{x}_r & -\dot{y}_r\dot{\theta}_g\dot{y}_r & -\dot{y}_r\dot{\theta}_g\dot{z}_r \\ \dot{x}_r\dot{\theta}_g\dot{x}_r & \dot{x}_r\dot{\theta}_g\dot{y}_r & \dot{x}_r\dot{\theta}_g\dot{z}_r \\ 0 & 0 & 0 \\ \dot{x}_r\dot{x}_r & \dot{x}_r\dot{y}_r & \dot{x}_r\dot{z}_r \\ \dot{y}_r\dot{x}_r & \dot{y}_r\dot{y}_r & \dot{y}_r\dot{z}_r \\ \dot{z}_r\dot{x}_r & \dot{z}_r\dot{y}_r & \dot{z}_r\dot{z}_r \end{bmatrix} \tag{8.6 -8}$$

and

$\frac{\partial \rho_D}{\partial \bar{x}_t}$ is the matrix containing the partial derivatives of the atmospheric density with respect to (see section 8.7.4 on atmospheric density partial derivatives). Because the density is not a function of satellite velocity, the required partials are $\frac{\partial \rho_D}{\partial \bar{r}}$.

8.7 ATMOSPHERIC DENSITY

In the computation of drag, it is essential to obtain models of the atmospheric density which will yield realistic perturbations due to drag. The GEODYN program uses the 1971 revised Jacchia Model which considers the densities between 90 km and 2500 km. The following discussion will cover primarily the assumptions of the models and empirical formulae used in the density subroutine. The procedure for empirically evaluating the density tables will also be included in the discussion.

8.7.1 Jacchia 1971 Density Model

The 1971 revised Jacchia model is based on Jacchia's 1971 report, "Revised Static Models of the Thermosphere and Exosphere with Empirical Temperature Profiles" [8 - 1]. The density computation from exospheric temperature as well as from variations independent of temperature is based on density data appearing in that report. This data presented in Table 8.7-1 shows the density distribution at varying altitudes and exospheric temperatures. For further detailed development of these empirical formulae, the interested reader should consult the aforementioned report and Jacchia's 1970 report [8 - 2].

8.7.1.1 The Assumptions of the Model The Jacchia 1971 model (J71) is based on empirically determined formulae with some inherent simplifying assumptions. Such an approach is taken primarily because the various processes occurring in different regions of the atmosphere are complex in nature. Moreover, at present, a thorough comprehension of such processes is lacking. The present J71 model is patterned after the J65a (Jacchia 1965a) model which was based upon previous assumptions by Nicolet [8 - 3].

In Nicolet's atmospheric model, temperature is considered as the primary parameter with all other physical parameters such as density and pressure derivable from temperature. This approach was adopted by Jacchia in his 165a model. However, in the J71 model, there are variations modeled by Jacchia which are independent of temperature. They are the semi-annual variations, seasonal- latitudinal variations of the lower thermosphere, and seasonal-latitudinal variations of helium, all of which involve a time dependency. Although in J71 Jacchia mentions variations in hydrogen. concentration, he does not attempt any quantitative evaluation of these variations.

Composition

The J71 model has assumed that the only constituents of the atmosphere arc nitrogen, oxygen, argon, helium, and hydrogen. This composition is assumed to exist in a state of mixing at heights below 100 km and in a diffusion state at higher altitudes. A further assumption on the composition of the atmosphere is that any variation in the mean molecular mass, M, in the mixing region is the result of oxygen dissociation only. The variation in M has been described by an empirical profile for heights ranging from 90 km to 100 km.

It is also believed that gravitational separation for helium exists at lower height than for the other components. To avoid the cumbersome ordeal of modeling a separate homopause for helium, Jacchia has modified the concentration at sea-level by a certain amount such that at altitudes where helium becomes a substantial constituent, the modeled densities will correspond to the observed densities. Although this will yield a higher helium density below 100 km, the contribution of helium to the overall density will be negligible below this height. Hydrogen does not become part of the density model until a height of 500 km. At this altitude, hydrogen is assumed to be in the diffusion equilibrium state.

Temperature

The temperature above the thermopause.is referred to as the exospheric temperature. Although this temperature is independent of height, it is subject to solar activity, geomagnetic activity, and diurnal and other variations. The J71 model assumes constant boundary conditions of 90 km with a constant thermodynamic temperature of 1830 K at this height. From numerous atmospheric conditions it is suggested that the atmospheric conditions at 90 km do indeed vary nominally, and thus, his assumption may be reasonably acceptable [8 - 4]. Profiles of the thermodynamic temperature show that it increases with height and reaches an inflection point at 125 km. Above this altitude, this temperature asymptotically attains the value of the exospheric temperature. An analytic model of the atmospheric densities by Roberts [8 - 4] has been constructed based on modifications to Jacchia's 1970 temperature profile between 90 km and 125 km, The J71 model assumes that the basic shape of the temperature profiles remains unchanged during atmospheric heating due to geomagnetic storms, In all likelihood, the profiles at low altitudes become distorted to yield higher temperatures during such occurrences. Since the J71 model assumes the atmosphere to be in static equilibrium, for any sudden changes in solar activity or in geophysical conditions, which are characteristically dynamic, the model will generally be unable to properly represent the variations in both temperature and density due to this invalid assumption.

The priority has been given to the best representation of density.

8.7.1.2 Variations in the Thermosphere and Exosphere Several types of variations occurring in the different regions of the atmosphere are incorporated in the J71 model. These variations are: solar activity variations, diurnal variations, geomagnetic activity variations, semi-annual variation, seasonal-latitudinal variations of the lower thermosphere, and seasonal-latitudinal variations of helium. Still another variation which is not quantitatively evaluated by J71 is the rapid density fluctuations believed to be associated with gravity waves [8 - 1]. Each of the above variations may be modeled empirically from observable data. However, because a static model is used, the various predictions will exhibit different degrees of accuracy for each variation. It is fundamental, however, to note that unless the characteristic time for which these variations occur is much longer than that for the processes of diffusion, conduction, and convection to occur, the predictions will be poor [8- 1].

Solar Activity

The variations in the thermosphere and exosphere as a result of solar activity are of a dual nature. One type of variation is a slow variation which prevails over an 11-year period as the average solar flux strength varies during the solar cycle. The other type is a rapid day-to-day variation due to the actively changing solar regions which appear and disappear as the sun rotates and as sunspots are formed.

To observe such activities, the 10.7 cm solar flux line is commonly used as an index of solar activity. The National Research Council in Ottawa has made daily measurements on this flux line since 1947. These values appear monthly in the "Solar Geophysical Data (Prompt Reports)" by the National Oceanic and Atmospheric Administration and the Environmental Data Service in Boulder, Colorado (U.S. Department of Commerce). A linear relationship exists between the average 10.7 cm flux and the average nighttime minimum global exospheric temperature (Jacchia 1971) and may be expressed as:

$$\bar{T}_{\infty} = 379^{\circ} + 3.24^{\circ}\bar{F}_{10.7}(^{\circ}Kelvin) \qquad (8.7\text{ -1})$$

where

$\bar{T}_{\infty}$ is the average nighttime minimum global exospheric temperature averaged over three solar rotations (81 days).

$\bar{F}_{10.7}$ is the average 10.7cm flux strength over three solar rotations and measured in units of 10^{-22} watts $m^{-2}Hz^{-1}$ bandwidth.

Equation (8.7-1) expresses the relationship with solar flux when the planetary geomagnetic index, K_p, is zero; i.e., for no geomagnetic disturbances. The nighttime minimum of the global exospheric temperature for a given day [8 - 1] is

$$\bar{T}_c = \bar{T}_\infty + 1.3^\circ(F_{10.7} - \bar{F}_{10.7}) \quad (8.7\text{ -}2)$$

where

$F_{10.7}$ is the daily value of the 10.7cm solar flux (in the same units as $\bar{F}_{10.7}$) for one day earlier, since there is a one day lag of the temperature variation response to the solar flux.

Thus, Equation (8.7-2) models a daily temperature variation about the average nighttime minimum global temperature as determined in Equation (8.7-1).

Diurnal Variations

Computations from drag measurements have indicated that the atmospheric density distribution varies from day to night. The densities arc at a peak at 2 P.M. local solar time (LST) approximately at the latitude of the sub-solar point, and at a minimum at 3 AM. (LST) approximately of the same latitude ill the opposite hemisphere. The diurnal variation of density at any point is subject to seasonal changes. By empirical relationships, this variation may be described in terms of the temperature. Again, because a static modeJ used, the accuracy of this temperature is open to question.

At a particular hour and geographic location, the temperature, T_l, can be expressed in terms of the actual global nighttime minimum temperature, T_c ,for the given day [8 - 1]. Thus, we may write

$$T_l = T_C(1 + R\sin^m\theta)\left[1 + R\frac{\cos^m\eta - \sin^m\theta}{1 + R\sin^m\theta}\right]\cos^n\frac{\tau}{2} \quad (8.7\text{ -}3)$$

where

R = 0.3

m = 2.2

n = 3.0

τ = $H + \beta + p\sin(H + \gamma)$ for ($-\pi < \tau < \pi$)

β = -37° (lag of the temperature maximum with the uppermost point of the sun.)

p = +6° (introduces an asymmetry in the temperature curve.)

γ = +43° (determines the location of the asymmetry in the temperature curve.)

$\eta = \frac{1}{2}$ABS $(\phi' - \phi_{\odot})$

$\theta = \frac{1}{2}$ABS$(\phi' - \phi_{\odot})$

ϕ' = geographic (geocentric) latitude

$\delta_{\odot}$ = declination of the sun

H = hour angle of the sun (when the point considered, the sun, and the earth's axis are all coplanar, $H = 0$. The hour angle is measured westward 0° to 360°)

The parameter R determines the relative amplitude of the temperature variation. Jacchia and his associates have undertaken investigations of R which reveal indications of its variation in time and with altitude. After consulting 1969-1970 data, Jacchia presently has abandoned any attempt at any definitive conclusions about the variations of R with time or with solar activity [8 - 1]. Instead, he believes this evidence to be the. result of inherent limitations of the static atmospheric representation. Consequently, in the J71 model, a constant value of $R = 0.3$ is maintained.

Geomagnetic Activity

Precise effects of geomagnetic activity cannot be obtained by present measurements from satellite drag, since such techniques can only show averaged values of densities. It is also realized that the consequences of a geomagnetic disturbance in view of the atmospheric temperatures and densities over the global regions are of a complex nature. However, when such disturbances occur, there are indications of increases in temperature and density in the thermosphere above the aurora belt. By the time this atmospheric disturbance reaches the equatorial regions, a period of roughly 7 hours, the effects are damped out considerably [8 - 1].

A static model description of temperature and density cannot be viewed accurately since the propagation time of the geomagnetic storms is relatively short. Therefore, any empirical formulae used to compute the effects on the parameters yield only a vague picture.

Jacchia et al (1967) [8 - 18] have related the exospheric temperature to the 3-hourly planetary geomagnetic index K_p. The quantity K, is used as a measure of a three-hour variation in the earth's magnetic field. The response of the temperature change to geomagnetic storms lags the variation in K_p by about 6.7 hours. In the following equation [8 - 1] the correction to the exospheric temperature due to geomagnetic activity is

$$\Delta T_{\infty} = 28^{\circ} K_p + 0.03^{\circ} exp(K_p) \quad (8.7\ \text{-4})$$

for heights above 200km.

Although this K_p in equation (8.7-4) is a three- hour planetary geomagnetic index, a K_p, averaged over a 24-hour period is used to minimize the amount of input data to GEODYN. The loss of accuracy in using the daily average of K, is minimized, since the above equation is for a smoothed effect of the variations derived from satellite data. Below 200 km, density predictions from equation (8.7-4) prove to be too low. Better results are obtained if the variations were represented as a two-step hybrid formula in which a correction to the density and to the temperature is made. Thus, in J71 the following hybrid formula [8 - 1) is given as

$$P_4 = \Delta \log_1 0\rho = 0.012K_p + 1.2 \times 10^{-5} exp(K_p) \quad (a)$$
$$\Delta T_\infty = 14^\circ K_p + 0.02^\circ exp(K_p) \quad (b) \qquad (8.7\ \text{-}5)$$

where $\Delta \log_{10} \rho$ is the decimal logarithm correction to the density ρ.

The values of a three-hour K_p, index are available along with the daily solar flux data in the publication "Solar Geophysical Data" by the N.O.A.A. and E.D.S., Boulder, Colorado (Department of Commerce). In computing the exospheric temperature, accurate daily values for both the solar and geomagnetic flux must be used. These values are updated as new information is received. This information may be updated using the appropriate GEODYN Input Cards. The user should be aware of the fact that these tables are expanded as new information is made available [8 - 3].

A midpoint point average is used to compute the six solar rotation flux values $\bar{F}_{10.7}$.

Semiannual Variation

The semiannual variation at present is least understood of the atmospheric variations. In past models, J65, attempts at empirically relating the temperature to this variation seemed appropriate in the range of heights, 250 to 650 krn, for which data was available. However, with the availability of new data for a wider range of altitudes, larger discrepancies in the densities appeared. After close scrutiny, Jacchia in 1971 [8 - 1] found that the amplitude of the semiannual density does not appear to be connected with the solar activity. It does, however, show a strong dependence all height and a variation from year to year. Drag analyses from the Explorer 32 satellite have revealed that a primary minimum in July and a primary maximum in October occur for the average density variation [8 - 1]. Jacchia in J7l expresses the semiannual variation as a product function [8 - 1] in which

$$P_2 = \Delta \log_{10} \rho = f(z)g(t) \qquad (8.7\ \text{-}6)$$

where $f(z)$ is the relationship between the amplitude, i.e., the difference between the primary maximum and minimum, and the height, z, and where $g(t)$ is the average density variation as a function of time for the amplitude normalized to 1. The two expressions for $f(z)$ and $g(t)$ which yield the best fit to the data are

$$f(z) = (5.876 \times 10^{-7} z^{2.331} + 0.06328) exp(-2.868 \times 10^{-3} z) \qquad (8.7\ \text{-}7)$$

for z in kilometers;

$$g(z) = 0.02835 + 0.3817[1 + 0.4671 \sin(2\pi\tau + 4.1370)] \sin(4\pi + 4.259) \qquad (8.7\ \text{-}8)$$

where

$\tau = \Phi + 0.09544[(0.5 + 0.5 \sin(2\pi\Phi + 6.035))^{1.650} - 0.5]$

$\Phi = \frac{(t-36204)}{365.2422}$

t =time expressed in Modifed Julian Days

(M. J. D. = Julian Day minus 2 400 000.5)

M. J. D. = 36204 is January 1, 1958.

The term Φ is the phase of the semiannual variation which is the number of days elapsed since January 1, 1958 divided by the number of days for the tropical year. Seasonal-Latitudinal Variations of the Lower Thermosphere In the lower thermosphere, from about 90 kill to 120 km, there are variations occurring in temperature and and density depending on the latitude and the season. Only the density variation is considered because any temperature variation proves to be too tedious to handle. Between the heights from 90 km to 100 km, there is a rapid increase in the amplitude of this variation in density with a maximum amplitude occurring between 105 and 120 km [8 - 1]. Above 120 km there is no data on which to base predictions of the seasonal-latitudinal variations. This variation appears to decrease in amplitude to the point where negligible fluctuations exist at 150 km. Therefore, the seasonal-latitudinal variations are neglected at heights above 160 km. Jacchia in J71 fits the seasonal variations to an empirical correction to the decimal logarithm of the density [8 - I] as follows:

$$P_3 = \Delta \log_{10} \rho = S \frac{\phi'}{|\phi'|} P \sin^2 \phi' \qquad (8.7\ \text{-}9)$$

where

Φ' $=$ geographic latitude

S $= 0.014\ (z-90)exp[-0.0013(z-90)^2]$

z $=$ height in kilometers

P $= \sin(2\pi\Phi + 1.72)$

Φ $=$ phase as given in equation (8.7-8).

Seasonal-Latitudinal Variations of Helium

Helium in the atmosphere has been observed to migrate towards the winter pole. The phenomenon of this seasonal shift in the helium concentration in the upper atmosphere is not yet understood. It therefore becomes necessary to perform an empirical fit to drag data from which this seasonal variation is derived. The expression which. is used in J71 [8 - 1] to describe the helium variation

$$Q_2 = \Delta \log_{10} n(He) = 0.65|\frac{\delta\odot}{\epsilon}|\left[\sin^3\left[\frac{\pi}{4} - \frac{\phi}{2}\frac{\delta\odot}{|\delta\odot\ |}\right] - \sin^3\frac{\pi}{4}\right] \qquad (8.7\ \text{-}10)$$

where

$n(He)$ $=$ number density of helium (number of particles/cm)

$\delta\odot$ $=$ declination of the sun

Φ $=$ geographic latitude

ϵ $=$ obliquity of the ecliptic ($\epsilon = 23.44°$)

The variation of the helium density in subroutine DENSTY is not considered for heights below 500 km. It is also neglected for latitudes whose absolute value is less than 15° between the range of heights from 500 km to 800 km.

The correction to the density due to the seasonal latitudinal variations of helium is then

$$\Delta\rho_D = 10^{\log_{10} n(He)}[10^{\Delta \log_{10} n(He)} - 1]C \quad gm/cm^3 \qquad (8.7\ \text{-}11)$$

where

C is the molecular mass of Helium divided by Avogadro's Number.

8.7.1.3 Polynomial Fit of Density Tables In Jacchia's 1971 report [8 - 1], there are tables showing the variation of density with altitude and exospheric temperature. From heights of 90 km to 100 km, the density values were obtained by numerically integrating the barometric equations. The diffusion equation was numerically integrated to obtain values of the density on the altitude range: 100 km $< Z \leq$ 2500 km. In both cases, an empirical temperature profile was used for each exospheric temperature. In the GEODYN the atmospheric density is computed based on the data from these tables after appropriate corrections are applied to the exospheric temperature. The tabulated densities have been fitted to various degree polynomials of the form;

$$P_1 = \log_{10} \rho_{DT} = \sum\nolimits_i h^{i-1} \sum\nolimits_j a_{ij} T^{j-1} \tag{8.7 -12}$$

where

ρ_{DT} is the density in g/cm^3

T is the exospheric temperature,

h is the spheroidal height (altitude), and

a_{ij} is a set of appropriate coefficients for the density tables.

A similar form of polynomial was used in fitting the helium number density.

$$Q_1 = \log_{10} n(He) = \sum_i h^{i-1} \sum_j b_{ij} T^{(j-1)} \tag{8.7 -13}$$

where

$\log_{10} n(He)$ is the Helium number density,

T is the exospheric temperature,

h is the spheroidal height, and

b_{ij} is a set of appropriate coefficients for the Helium number density tables.

Because the tabulated density values incorporate a wide range of exospheric temperatures ($500° - 1900°K$) and altitude (90 - 2500 km), it is difficult to obtain an accurate representation from a single polynomial of the form in equation (8.7-11). Therefore, appropriate coefficients and polynomials were selected accordingly for varying height and temperature ranges. The Jacchia density tables were separated into seven sections for polynomial fitting in order to yield reasonable results.

The lowest region (90 - 200 km) is expressed by a third degree polynomial in height with its coefficients also fitted by a third degree polynomial in temperature from $500° - 1900°$K.

For the regions 200 - 500 km, 500 - 1000 km, and 1000 -2500 km third degree polynomials in altitude were used. Within these regions height coefficients were fitted to two ranges of temperatures, one from 500° − 800°K and the other from 800° − 1900°K. In the last region (1000 - 2500 krn) for the temperature range of 800° − 1900° K, the coefficients in height were fitted to a fourth degree polynomial temperature. In all other cases, the height coefficients were of a third degree fit. The coefficients for the selected polynomials for the total density are shown in Table 8.7-1. In Table 8.7-2, coefficients of polynomials for the helium number density are presented. The computed densities from the fitted polynomials show a reasonable percentage error from the Jacchia density tables. For each of the regions and temperature ranges, the maximum errors are given in Table 8.7-3. The largest error of 12% occurs in the region between 500 -1000 km in the temperature range of 500° − 800° K. In the region of 1000 - 2500 km with temperatures between 800° − 1900° K a fourth degree fit to the temperature yields a maximum error of 11.0% in the densities.

The helium number density fits are also given in Table 8.7-3. As one can see, the values of the number density are quite satisfactorily fitted by the polynomials. The maximum error in the whole range of heights and temperatures is only 2.8%.

Overall, these fits could be improved by either using higher degree polynomials or possibly other functions, or by further sub-dividing the density table. However, these maximum errors appear to be tolerable since they are considered to be within the range of accuracy of the model presently used. Above 2500 km, the density was found to be negligibly small, and therefore, was set to zero.

Table 1. Table 8.7-1 Density Polynomial Coefficients (For Decimal Logarithm of Density)

	T0	T1	T2	T3	T4
90-200 KH					
h0	0.42208SE+01	0.983929E-02	-.649521E-05	0.147153E-08	
h1	-.201342E+00	-.234119E-03	0.153367E-06	-.346749E-10	
h2	0.785919E-03	0.169661E-05	-.110604E-08	0.250069E-12	
h3	-.120874E-05	-.343600E-08	0.224571E-11	-.510691E-15	
200-500 KH FOR 500-800°K					
h0	-.128384E+02	0.407090E-02	0.970741E-05	-.106425E-07	
h1	0.822821E-01	-.312150E-03	0.265426E-06	-.551933E-10	
h2	-.689505E-03	0.241015E-05	-.270581E-08	0.990035E-12	
h3	0.112628E-05	-.418072E-08	0.506171E-11	-.204837E-14	
200-500 KH FOR 800-1900°K					
h0	-.845952E+01	-.150005E-03	-.626402E-06	0.246119E-09	
h1	-.283949E-01	0.177601E-06	0.613977E-08	-.233615E-11	
h2	0.559979E-05	0.774612E-07	-.594920E-10	0.149206E-13	
h3	0.394342E-08	-.764352E-10	0.583326E-13	-.145952E-16	
500-1000 KM FOR 500-800°K					
h0	-.776587E+02	0.167271E+00	-.565702E-04	-.504240E-07	
h1	0.306381E+00	-.989364E-03	0.749320E-06	-.531783E-10	
h2	-.389349E-03	0.129726E-05	-.107765E-08	0.141909E-12	
h3	0.159620E-06	-.540490E-09	0.467086E-12	-.718865E-16	
500-1000 KM FOR 800-1900°K					
" h0	0.500815E+02	-.126002E+00	0.838957E-04	-.182762E-07	
h1	-.305716E+00	0.617057E-03	-.414429E-06	0.910959E-10	
h2	0.417666E-03	-.887429E-06	0.610397E-09	-.136339E-12	
h3	-.179649E-06	0.393863E-09	-.276393E-12	0.626491E-16	
1000-2500 KM FOR 500-800°K					
h0	0.365321E+02	-.261563E+00	0.419634E-03	-.216612E-06	
h1	-.483521E-01	0.268015E-03	-.482137E-06	0.270953E-09	
h2	0.111413E-04	-.774998E-07	0.160423E-09	-.990550E-13	
h3	-.250589E-10	0.447248E-11	-.140854E-13	0.104429E-16	
1000-1250 KM FOR 800-1900°K					
h0	0.524100E+02	-.206525E+00	0.216416E-03	-.906229E-07	0.130541E-10
h1	-.143546E+00	0.431133E-03	-.461365E-06	0.201789E-09	-.308883E-13
h2	0.876930E-04	-.271569E-06	0.297453E-09	-.134250E-12	0.213697E-16
h3	-.157162E-07	0.496307E-10	-.552974E-13	0.254318E-16	-.413038E-20

Table 2. Table 8.7-2 Helium Density Polynomial Coefficients (Decimal Log of Helium Number Density)

	T0	T1	T2	T3	T4
500-1000 KM FOR 500-800°K					
h0	0.937121E+01	-.526336E-02	0.529828E-05	-.204706E-08	
h1	-.131408E-01	0.312183E-04	-.325977E-07	0.125725E-10	
h2	0.260710E-05	-.757300E-08	0.930576E-11	-.406694E-14	
h3	-.521562E-09	0.190563E-11	-.265782E-14	0.125349E-17	
500-1000 KM FOR 800-1900°K					
h0	0.839138E+01	-.164334E-02	0.780320E-06	-.143227E-09	
h1	-.690494E-02	0.841383E-05	-.445772E-08	0.856271E-12	
h2	0.105103E-05	-.126629E-08	0.711337E-12	-.141795E-15	
h3	-.122221E-09	0.147453E-12	-.976579E-16	0.214583E-19	
1000-2500 KM FOR 500-800°K					
h0	0.910459E+01	-.434096E-02	0.402920E-05	-.145215E-08	
h1	-.122592E-01	0.279509E-04	-.279720E-07	0.103710E-10	
h2	0.158930E-05	-.358626E-08	0.354761E-11	-.129853E-14	
h3	-.118288E-09	0.261383E-12	-.252266E-15	0.897136E-19	
1000-2500 KM FOR 800-1900○K					
h0	0.861203E+01	-.253633E-02	0.189793E-05	-.736962E-09	0.113880E-12
hI	-.848471E-02	0.140837E -04	-.113857E-07	0.448705E-11	-.690642E-15
h2	0.115432E-05	-.198839-08	0.166354E-11	-.676278E-15	0.107063E-18
h3	-.945207E-10	0.173870E-12	-.153679E-15	0.654017E-19	-.107605E-22

Table 3. Table 8.7-3 Percentage Error of Polynomial Fits to the Densities

Height Range	Temperature Range	Maximum Percent Error	
(KN)	(Degrees Kelvin)	Total Density	Helium Density
90-200	500-1900	11.0	
200-500	500-800	11. 6	
200-500	800-1900	5.13	
500-1000	500-800	12.0	0.44
500-1000	800-1900	8.85	2.8
1000-2500	500-800	4. 1	1. 0
1000-2500	800-l900	11.0	1. 25

8.7.1.4 The Density Computation When all of the terms contributing to the atmosphere density are combined

$$\rho_D = 10^3[10^{P_1+P_2+P_3+P_4} + 10^{Q_1}(10^{Q_2} - 1)C] \qquad (8.7\text{ -14})$$

where

ρ_D the atmospheric density in Kg/m^3

P_1 is given by equation (8.7-12),

P_2 is given by equation (8.7-6),

P_3 is given by equation (8.7-9),

P_4 is given by equation (8.7-5a),

Q_1 is given by equation (8.7-13),

Q_2 is given by equation (8.7-10), and

C is the molecular mass of Helium divided by Avogadro's Number = 0.6646 (10^{23})

8.7.1.5 Density Partial Derivatives In addition to the density, GEODYN also requires the partial derivatives of the density with respect to the Cartesian position coordinates. These partials are used in computing the drag contributions to the variational equations.

The spatial partial derivatives of the atmospheric density are

$$\frac{\partial \rho_D}{\partial \bar{r}} = \frac{\partial \rho_D}{\partial \phi}\frac{\partial \phi}{\partial \bar{r}} + \frac{\partial \rho_D}{\partial \lambda}\frac{\partial \lambda}{\partial \bar{r}} + \frac{\partial \rho_D}{\partial h}\frac{\partial h}{\partial \bar{r}} \qquad (8.7\text{ -15})$$

where

h = spheroid height of the satellite

ϕ = sub-satellite latitude

λ =sub-satellite longtitude

$\bar{r}$ = true of date position vector of the satellite

Variations in atmospheric density are primarily due to changes in height. Therefore, only height variations are computed by GEODYN and

$$\frac{\partial \rho_D}{\partial \phi} = 0 \quad (8.7\text{-}16)$$

$$\frac{\partial \rho_D}{\partial \lambda} = 0$$

and consequently

$$\frac{\partial \rho_D}{\partial \bar{r}} = \frac{\partial \rho_D}{\partial h}\frac{\partial h}{\partial \bar{r}} \quad (8.7\text{-}17)$$

where $\frac{\partial h}{\partial \bar{r}}$ is presented along with the spheroid height computation in Section 5.1.

The density is given (Section 8.7.4) by

$$\rho_D = 10^3[10^{P_1+P_2+P_3+P_4} + 10^{Q_1}(10^{Q_2} - 1)C] \quad (8.7\text{-}18)$$

where

$$\rho_D = \text{density in } Kg/m^3$$

$$P_1 = \sum_{i=1}^{n} h^{(i-1)} \sum_{j=1}^{n} a_{ij} T^{(j-1)} \quad (8.7\text{-}19)$$

$$P_2 = g(t)[5.876(10^{-7})h^{2.331} + 0.06328]exp[-2.868(10^{-3})h] \quad (8.7\text{-}20)$$

$$P_3 = 0.014(h-90)P\frac{\phi}{|\phi|}\sin^2\phi' exp[-0.0013(h-90)^2] \quad (8.7\text{-}21)$$

$$P_4 = 0.012K_P + 1.2(10^{-5})exp(K_\rho) \quad (8.7\text{-}22)$$

$$Q_1 = \sum_{i-1}^{n} h^{(i-1)} \sum_{j=1}^{m} b_{ij} T^{(j-1)} = \log_{10} n(He) \quad (8.7\text{-}23)$$

$$Q_2 = \Delta \log_{10} n(He) \quad (8.7\text{-}24)$$

C = the molecular mass of Helium divided by Avogadro's Number.

h = height in Km.

a_{ij} = polynomial coefficients used to fit the density table.

b_{ij} = polynomial coefficients used to fit the Helium number density table.

All other terms are defined in Section 8.7.1.4; and need no further clarification at this point since they are constants in the partial derivative equations.

Defining two basic derivative formulae,

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{du(x)}{dx} \tag{8.7 -25}$$

$$\frac{d}{dx}10^{u(x)} = \frac{d}{dx}e^{u(x)\ln 10} \tag{8.7 -26}$$

$$= \ln 10 e^{u(x)\ln 10}\frac{du(x)}{dx} = 10^{u(x)}\ln 10\frac{du(x)}{dx}$$

And it follows that

$$\frac{\partial}{\partial h}10^{P_1+P_2+P_3+P_4} = 10^{P_1+P_2+P_3+P_4}\ln 10\frac{\partial}{\partial h}(P_1+P_2+P_3+P_4) \tag{8.7 -27}$$

$$\frac{\partial}{\partial h}10^{Q_1} = 10^{Q_1}\ln 10\frac{\partial Q_1}{\partial h} \tag{8.7 -28}$$

Differentiating the components of (8.7-27) and (8.7-28)

$$\frac{\partial P_1}{\partial h} = \sum_{i=1}^{n}(i-1)h^{(i-2)}\sum_{j=1}^{m}a_{ij}T6(j-1) \tag{8.7 -29}$$

$$\begin{aligned}\frac{\partial P_2}{\partial h} &= g(t)(5.876(10^{-7})(2.331)h^{1.331}exp[-2.868(10^{-3})h] \\ &+[5.876(10^{-7})h^{2.331}+0.06328](-2.868)(10^{-3})exp[-2.868(10^{-3})h])\end{aligned} \tag{8.7 -30}$$

$$\frac{\partial P_3}{\partial h} = 0.014P\frac{\phi'}{|\phi|}sin^2\phi' exp[-0.0013(h-90)^2](1+2(h-90)^2(-0.0013)) \tag{8.7 -31}$$

$$\frac{\partial P_4}{\partial h} = 0 \tag{8.7 -32}$$

$$\frac{\partial Q_1}{\partial h} = \sum_{i=2}^{n}(i-1)h^{(i-2)}\sum_{j=1}^{m}a_{ij}T^{(j-1)} \tag{8.7 -33}$$

The resulting partials are in the units of (Kg/m^3)/Km and must therefore be multiplied by 10^{-3}.

$$\frac{\partial \rho_D}{\partial h}\frac{\partial}{\partial h}10^{P_1+P_2+P_3+P_4} + (10^{Q_2}-1)C\frac{\partial}{\partial h}10^{Q_1} \tag{8.7 -34}$$

The units of (8.7-34) are then (Kg/nt^4).

8.7.2 Thermospheric Drag Model (DTM)

DTM is an empirical dimensional model (thermospheric temperature density and composition) developed in terms of spherical harmonics by using satellite drag data which cover almost two solar cycles. The model gives total densities, partial densities and temperature as a function of solar and geomagnetic activity, local time, day of the year, altitude and latitude. The DTM like the Jacchia model is based on satellite drag data combined with the assumption of diffusive equilibrium. Furthermore, the existence of a new global exospheric model based on direct optical determinations of temperature, provides a different interpretation of the drag data.

8.7.2.1 Assumptions of the Model The majority of the satellite drag data are obtained in height region 200 km and 1200 km. By numerical integration or analytical expressions, it is possible to compute total atmospheric density in the vicinity of the satellite perigee. As a consequence of diffusive separation above 1000 km altitude, molecular nitrogen N_2, atomic oxygen O and helium He, are the major atmospheric constituents. Therefore the total density can be observed as densities of these individual constituents. Molecular oxygen O_2 and hydrogen H are not major components within the same range of height and we must introduce some assumptions for their vertical distribution. Furthermore a horizontal and vertical distribution must be adopted to compute distribution of the individual components. Temperature T is given as a function of z (satellite altitude above the reference ellipsoid).

$$T(z) = T_\infty - (T_\infty - T_{120}) exp(-\sigma\zeta) \qquad (8.7\ \text{-}35)$$

where

T_∞ is the thermopause temperature

T_{120} is the constant temperature of 120 km (lower boundary altitude) = $3800°K$

ζ is the geopotential altitude given by:

$$\zeta = \frac{(z-120)(a+120)}{a+z} \qquad (8.7\ \text{-}36)$$

where

a is the semi-major axis of the reference ellipsoid (6356.77 km)

$\sigma = S + (a+120)^{-1}$

$S = 0.02$ (temperature gradient parameter)

S and T_{120} are used in the model as constants although they actually vary with seasons. The contribution of hydrogen H to the total density is negligible for the considered range of heights. However, for the determination of the helium (He) density above 500 km, we subtract the atomic hydrogen effect.

For molecular oxygen (O_2) contribution to the total density is less than 5%. Therefore, they adopted a constant value of $4.75 \times 10^{10} cm^{-3}$ at the lower boundary of 120 km.

8.7.2.2 Mathematical Formulation The various densities contributing to the total density are given by:

$$n_i(z) = A_{1_i} exp[G_i(L) - 1] f_i(z) \quad (8.7\ \text{-37})$$

where

A_{1_i} is a constant

$f_i(z)$ is a function which result from the integration of the diffusive equilibrium with temperature profile given by (8.7-35)

$$f_i(z) = \left[\frac{1-a}{1-ae^{\sigma\zeta}}\right]^{1+a_i+\gamma_i} exp[-\sigma\gamma_i\zeta] \quad (8.7\ \text{-38})$$

where

a_i = thermic diffusion factors (constant for each constituent)

$\gamma_i = \frac{(m_i g_{z_0})}{\sigma K T_\infty}$

$a = \frac{T_\infty - T_{z_0}}{T_\infty}$

m_i = molecular mass of constituent

g_{z_0} = gravity at z_0

K = Boltzman Constant

$T_\infty = A_1 G(L)$

$G_i(L)$ from spherical harmonic development depends on various parameters contributing to the density variation

Omitting the subscript i , the function $G(L)$ is given by:

$$G(L) = 1 + F_1 + M + \sum_{q=1}^{\infty} a_q^0 P_q^0(\theta) + \beta \sum_{p=1}^{\infty} b_p^0 P_p^0(\theta) \cos[p\Omega(d - \delta_p)] + \beta \sum_{n=1}^{\infty} \sum_{m=1}^{n} [c_n^m P_n^m(\theta) \cos(m\omega t) + d_n^m P_n^m(\theta) \sin(m\omega t)] \quad (8.7\text{ -}39)$$

where

$\Omega = \frac{2\pi}{365}(\text{day}^{-1})$

$\omega = \frac{2\pi}{24}(\text{hour}^{-1})$

P_n^m = associated Legendre functions described explicitly later

a, b, c, d = thermonical coefficient provided by the model

Solar activity effect:

$$F_1 = A_4(F - \bar{F}) + A_5(F + \bar{F})^2 + A_6(\bar{F} - 150) \quad (8.7\text{ -}40)$$

where

F = solar flux of the previous day

$\bar{F}$ = moving average of solar flux

Geomagnetic effect:

$$M = (A_7 + A_3 P_2^0) K_p \quad (8.7\text{ -}41)$$

where

K_p = geomagnetic index of 6 hours earlier

$P_2^0 = \frac{1}{2}(3 \sin\phi - 1)$

ϕ = geodetic latitude

Zonal latitude dependent term:

$$\sum_{q=1}^{\infty} a_q^0 P_q^0(\theta) \simeq A_2 P_2^0 + A_3 P_4^0 \quad (8.7\text{ -}42)$$

where

$$P_2^0 = \frac{1}{2}(3\sin^2\phi - 1)$$
$$P_4^0 = \frac{1}{8}(35\sin^4\phi - 30\sin^2\phi + 3)$$

Annual and semi-annual terms:

$$\sum_{p=0}^{\infty} b_p^0 P_p^0(\theta)\cos[p\Omega(d-\delta_p)] \simeq AN1 + AN2 + SAN1 + SAN2 \qquad (8.7\ \text{-}43)$$

where $AN1$, $SAN1$ are terms independent of sign of latitude and $AN2$, $SAN2$ are terms where the latitude has opposite signs in opposite hemispheres.

$$AN1 = (A_9 + A_{10}P_2^0\cos[\Omega(d - A_{11})] \qquad (8.7\ \text{-}44)$$
$$SAN1 = (A_{12} + A_{13}P_2^0\cos[2\Omega(d - A_{14})] \qquad (8.7\ \text{-}45)$$
$$AN2 = (A_{15}P_1^0 + A_{16}P_3^0 + A_{17}P_5^0\cos[\Omega(d - A_{18})] \qquad (8.7\ \text{-}46)$$
$$SAN2 = A_{19}P_1^0\cos[2\Omega(d - A_{20})] \qquad (8.7\ \text{-}47)$$

where

$$P_1^0 = \sin\phi$$
$$P_3^0 = \frac{1}{2}(5\sin^2\phi - 3)\sin\phi$$
$$P_5^0 = \frac{1}{8}(63\sin^4\phi - 70\sin^2\phi + 15)\sin\phi$$

Diurnal, Semidiurnal and Terdiurnal terms:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}[c_n^m P_n^m(\theta)\cos(m\omega t) + d_n^m P_n^m(\theta)\sin(m\omega t)] \cong D + SD + TD \qquad (8.7\ \text{-}48)$$

where

$$
\begin{aligned}
P_1^1 &= \cos\phi \\
P_2^1 &= \frac{3}{2}\sin\phi \\
P_3^1 &= \frac{3}{2}(5\sin^2\phi - 1)\cos\phi \\
P_5^1 &= \frac{1}{8}(315\sin^4\phi - 210\sin^2\phi + 15)\cos\phi \\
SD &= [A_{31}P_2^2 + A_{32}P_3^2\cos(\Omega(d - A_{18})]\cos 2\omega t \\
&\quad + [A_{33}P_2^2 + A_{34}P_3^2\cos(\Omega(d - a_{18}))]\sin 2\omega t
\end{aligned} \tag{8.7 -49}
$$

where

$$
\begin{aligned}
P_2^2 &= 3\cos^2\phi \\
P_3^2 &= 15\sin\phi\cos^2\phi \\
TD &= A_{35}P_3^3\cos 3\omega t + A_{36}P_3^3\sin 3\omega t
\end{aligned} \tag{8.7 -50}
$$

where

$$P_3^3 = 15\cos^3\phi$$

β in equation(8.7-39) is

$\beta = 1 + F_1$ for O, N_2, and T_∞ and
$\beta = 1$ for He

Finally, the total density, ρ. is given by the following equation

$$\rho = \sum_i n_i m_i \tag{8.7 -51}$$

where m_i = molecular mass of constituent i.

Finally, the acceleration due to drag is given by:

$$\ddot{\bar{X}}_D = -S \times \rho \times \bar{V}_{REL} \tag{8.7 -52}$$

where $\bar{V}_{REL}$ is the relative velocity of the satellite with respect to the atmosphere. In GEODYN II, we assume that the atmosphere is rotating with the same velocity as the Earth (rigid body).

$$S = 0.5 \times C_d \times \frac{A}{m} \tag{8.7 -53}$$

where

A = the cross sectional area of the satellite

m = the mass of the satellite, and

C_d = the Drag coefficient

8.8 TIDAL POTENTIALS

8.8.1 The Love Model

The gravitational potential originating from solid earth tides caused by a single disturbing body is given [8 - 11].

$$\begin{aligned} U_X(r) &= \frac{k_2}{2}\frac{GM_d}{R_d^3}\frac{R_e^5}{r^3}[3(\hat{R}_d \cdot \hat{r}) - 1] \\ &= \frac{k_2}{2}\frac{GM_d}{R_d^3}\left[\frac{M_d}{M_e}\frac{R_e}{R^d}\right]^3\left[\frac{R_e}{r}\right]^3[3(\hat{R}_d \cdot \hat{r})^2 - 1] \end{aligned} \tag{8.8 -1}$$

and the resultant acceleration on a satellite due to this potential is

$$\nabla U_D = \frac{k_2}{2}\frac{GM_d}{R_d^3}\frac{R_e^5}{r^3}([3 - 15(\hat{R}_d \cdot \hat{r})^2]\hat{r} + 6(\hat{R}_d \cdot \hat{r})\hat{R}_d) \tag{8.8 -2}$$

where

k_2 is the tidal coefficient of degree 2 called the "Love Number"

G is the universal gravitational constant

M_e is the mass of the earth

R_e is the mean earth radius

M_d is the mass of the disturbing body

M_e is the mass of the earth

R_d is the distance from COM_e^2 to COM_d^3

r is the distance from COM_e to the satellite

$\hat{R}_d$ is the unit vector from COM_e to COM_d

$\hat{r}$ is the unit vector from COM_e to satellite

8.8.2 The Expanded Tide Model

The gravitational potential arising from solid earth tides caused by a single disturbing body is given by (Kaula, 1969) [8 - 19, 8 - 20, 8 - 21]:

$$T_{2mp0} = A^*_{2mp0} \frac{P_{2m}(\sin(\frac{z}{r}))}{r^3} \left[\begin{bmatrix} \Gamma_+ \\ \Gamma_- \end{bmatrix}_{2mp0} \cos m(\frac{y}{x}) \pm \begin{bmatrix} \Gamma_- \\ \Gamma_+ \end{bmatrix}_{2mp0} \sin m(\frac{y}{x}) \right] \begin{cases} \text{upper line for } 2-m \text{ even} \\ \text{lower line for } 2-m \text{ odd} \end{cases}$$

where

$$\begin{aligned} A^*_{2mp0} &= \frac{GM^*}{a^{*3}} a_e^5 \frac{(2-m)!}{(2+m)!}(2-\delta_{om})G_{2po}(e^*) \\ \begin{bmatrix} \Gamma_+ \\ \Gamma_- \end{bmatrix}_{2mp0} &= K_{2mp0} F_{2mp}(i^*) \begin{bmatrix} \cos \\ \sin \end{bmatrix} (V^*_{mp} - \eta_{2mp0}) \\ V^*_{mp} &= (2-2p)(\omega^* + M^*) + m\Omega^* \end{aligned}$$

and

K_{2mp0} is the tidal coefficient of degree 2

G is the universal gravitational constant

M^* is the mass of the disturbing body

$a^*, e^*, i^*, l^*, \omega^*, \Omega^*$ are the Keplerian elements of the disturbing body

a_e is the radius of the earth

P_{2m} is the associated Legendre polynomial of degree 2

x, y, z are the Cartesian coordinates of the satellite

The resultant acceleration is then

$$\ddot{x} = -\sum_{m,p} \nabla T_{2mp0}$$

$$\nabla T_{2mp0} = -A^*_{2mp0}\left[\nabla\left[\frac{P_{2m}}{r^3}\right]\left[\begin{bmatrix}\Gamma_+\\ \Gamma_-\end{bmatrix}_{2mp0}\cos m(\frac{y}{x}) \pm \begin{bmatrix}\Gamma_+\\ \Gamma_-\end{bmatrix}_{2mp0}\sin m(\frac{y}{x})\right]\right.$$
$$\left.+\frac{P_{2m}}{r^3}\left[\begin{bmatrix}\Gamma_+\\ \Gamma_-\end{bmatrix}_{2mp0}\nabla\cos m(\frac{y}{x}) \pm \begin{bmatrix}\Gamma_+\\ \Gamma_-\end{bmatrix}_{2mp0}\nabla\sin m(\frac{y}{x})\right]\right]$$

Now further the variational equations are (summation over m and p implied):

$$\frac{\partial}{\partial \epsilon_{2mp0}}(-\nabla T_{2mp0}) = -A^*_{2mp0}\left[\nabla\left[\frac{P_{2m}}{r^3}\right]\left[\begin{bmatrix}\Gamma_-\\ -\Gamma_+\end{bmatrix}_{2mp0}\cos m(\frac{y}{x}) \pm \begin{bmatrix}-\Gamma_-\\ \Gamma_+\end{bmatrix}_{2mp0}\sin m(\frac{y}{x})\right]\right.$$
$$\left.+\frac{P_{2m}}{r^3}\left[\begin{bmatrix}\Gamma_-\\ -\Gamma_+\end{bmatrix}_{2mp0}\nabla\cos m(\frac{y}{x}) \pm \begin{bmatrix}-\Gamma_+\\ \Gamma_-\end{bmatrix}_{2mp0}\sin m(\frac{y}{x})\right]\right]$$

$$\frac{\partial}{\partial K_{2mp0}}(-\nabla T_{2mp0}) = \frac{\ddot{x}}{K_{2mp0}}$$

Comparison

Current Tide Model in GEODYN : truncated harmonics at $l = 2, m = p = q = 0$.
Expanded Tide Model : expanded harmonics up to $l = 2, p = 0, 1, 2, m = 0, 1, 2$ and $q = 0$.

8.8.3 The 8906 Tide Model

The GEODYN tide model, described by Colombo (1984) [8 - 34], corrects several problems which previously existed in the tidal force model, making GEODYN better suited to perform precise tidal analysis of tracking and altimetry data. Specifically, it includes the osculating eccentricity and inclination of the disturbing body in the ocean tide formulation. It also implements a side band modulation of the main tidal lines. In the process of making these modifications, every attempt has been made to improve the speed and accuracy of the computations. Christodoulidis et al. (1988) expresses the second degree tidal potential as:

$$V(\phi, \lambda, r) = \sum_f k_{2,f}\bar{A}_f G_D \frac{3-m}{3}(\frac{R}{r})^3 P_{2m}(\sin\phi)\cos\alpha_f^{SE} \qquad (8.8\text{ -3})$$

for the solid Earth and by

$$U(\phi, \lambda, r) = \sum_f \sum_{l,q,\pm} 4\pi G R_{\rho 0} (\frac{1+k_l}{2l+1}) C_{l,1,f^\pm} \cdot (\frac{R}{r})^{i+1} P_{l,q}(\sin\phi) \cos\alpha_{l,q,f^\pm} \qquad (8.8\text{ -4})$$

for the oceans, where the angular arguments are, respectively,

$$\alpha_f^{SE} = (\mp)\left[(2-2h)\omega^* + (2-2h+j)M^* + k\Omega^*\right] + m\Theta_g + m\lambda + \pi - m\frac{\pi}{2} + \delta_{2,f} \qquad (8.8\text{ -5})$$

and

$$\alpha_{l,q,f}^{\mp} = (\mp)\left[(2-2h)\omega^* + (2-2h+j)M^* + k\Omega^*\right] + m\Theta_g \pm q\lambda + \pi - m\frac{\pi}{2} + \zeta_{l,q,f^\mp} \qquad (8.8\text{ -6})$$

where

ϕ, λ, r latitude, east latitude, and radial distance of the point of evaluation;

Θ_g Greenwich sidereal hour angle;

$P_{l,q}(\sin\phi)$ associated Legendre functions

a^*, e^*, i^* Keplerian elements of the disturbing body

Ω^*, ω^*, M^* referred to the ecliptic;

a_m, e_m, i_m Keplerian elements of the Moon referred to

Ω_m, ω_m, M_m the ecliptic;

a_s, e_s, i_s Keplerian elements of the Sun referred to

Ω_s, ω_s, M_s the ecliptic;

μ_m gravitational constant times the mass of the Moon;

μ_s gravitational constant times the mass of the Sun;

μ gravitational constant times the mass of the Earth;

R average radius of the Earth;

G_D equivalent of the Doodson constant;

$\sum_f$ summation over all tide constituents if) in the expression of the tide-generating potential;

$\bar{A}_f$ equivalent of the Doodson coefficient;

$k_{2,f}$ second degree love number and is its phase;

k_l^l load deformation coefficients;

ρ_0 average density of the oceanic water;

$C_{l,q,f^\pm}, \zeta_{i,q,f^\pm}$ amplitude and phase in the spherical harmonic expansion of the ocean tides specified by I, q, and the tide constituent f.

Notice that the solid Earth potential incorporates the osculating inclination and eccentricity of the disturbing body in the Doodson coefficient. However, no term is a dependent upon these parameters in the ocean tide formulation. The lunar eccentricity can vary as much as 40% with respect to its mean value over the course of a month. The tidal amplitude exhibits a dependance approximately proportional to the J power of eccentricity, where J is the integer multiplier of longitude of perigee. For most major tides, where J = 0, this effect can be ignored. However, for J = 1 tides (L_2, N_2) and J = 2 tides (2 N_2) the variation in the tidal amplitude becomes significant.

$$\Delta\text{amplitude} \simeq \begin{matrix} 0\%(J=0) \\ 40\%(J=1) \\ 80\%(J=2) \end{matrix}$$

To correct this inconsistency, the new ocean tide model incorporates the osculating elements of the disturbing body as in the following:

$$U(\phi,\lambda,r) = \sum_f \sum_{l,q,\pm} \frac{V_f(t)}{\bar{V}_f} 4\pi G R_{\rho_0} (\frac{1+k_l}{2l+1}) \cdot C_{l,q,f^\pm} (\frac{R}{r})^{i+1} \bar{P}_{l,q}(\sin\phi) \cos\alpha_{l,q,f^\pm} \tag{8.8 -7}$$

where

$$V_f = A_f G_D$$

and where

$$\bar{V}_f = \bar{A}_f \bar{G}_D$$

The time varying tidal amplitude, V_f , incorporates the osculating eccentricity and inclination of the disturbing body. This expression is scaled by $\bar{V}_f$ to ensure that the value is dose to unity. The following efficiency features have also been added:

- Linearly interpolate tidal amplitude over a variable interval (TIDCON)

- Only compute distinct frequencies (TIDACC)
- Only compute distinct $q\lambda$ terms (TlDACC)

The new tidal model also allows for the scaling of the sidebands in conjunction with the adjustment of the main line constituents. Since the side bands are grouped in a band about the main line, the admittance function is linear and the tidal potential becomes, from equation (8.8.3-5)

$$U(\phi,\lambda,r) = \sum_f \sum_{l,q,\pm} 4\pi G R_{\rho_0} (\frac{1+k_i}{2l+1}) C_{l,q,f^{\pm}} \cdot \bar{P}_{l,q}(\sin\phi) \begin{cases} A_f(t)\cos(\sigma_{f_0} \pm q\lambda + \pi - m\frac{\pi}{2} + \epsilon) \\ +B_f(t)\sin(\sigma_{f_0} \pm q\lambda + \pi - m\frac{\pi}{2} + \epsilon) \end{cases} \tag{8.8 -8}$$

where

$$A_f(t) = \sum_j \frac{V_{f_f}(t)}{\bar{V}_{f_0}} \cos(\sigma_{f_j(t)} - \sigma_{f_0}(t)) \tag{8.8 -9}$$

and

$$B_f(t) = \sum_j \frac{V_{f_f}(t)}{\bar{V}_{f_0}} \sin(\sigma_{f_j(t)} - \sigma_{f_0}(t)) \tag{8.8 -10}$$

The code for equations (8.8.-7) and (8.8.-8) is primarily located in subroutine TIDACC, where the time dependent tidal accelerations are computed. Once per global iteration initializations are performed in PRETJD and PRESET. The Doodson coefficient is computed in ADOOD. Equations (8.8.-9) and (8.8.-10) are computed in subroutines AANDBE and AANDBO.

8.9 GENERAL ACCELERATIONS

GEODYN is capable of applying and solving for general satellite accelerations of the form

$$\ddot{\bar{X}} = \alpha\hat{u} \tag{8.9 -1}$$

where $\hat{u}$ is a unit vector defining the direction of the acceleration and α is the solved-for parameter. Below, let $\bar{X}$ and $\bar{V}$ be the satellite's true of date position and velocity.

Radial Acceleration

$$\hat{u} = \frac{\bar{X}}{|\bar{X}|} \tag{8.9 -2}$$

Cross-track Acceleration

$$\hat{u} = \frac{\bar{X} \times \bar{V}}{|\bar{X} \times \bar{V}|} \tag{8.9 -3}$$

Along track Acceleration

$$\hat{u} = \frac{\bar{V}}{|\bar{V}|} \tag{8.9 -4}$$

In-plane Acceleration

$$\hat{u} = \frac{(\bar{X} \times \bar{V}) \times \bar{X}}{|(\bar{X} \times \bar{V}) \times \bar{X}|} \tag{8.9 -5}$$

In order to define the GPS general accelerations the GPS body fixed coordinates must be defined.
let

$\bar{SS}$ == The vector from the satellite center of gravity (CG) to the sun's CG.

$\bar{R}$ == The vector from the center of earth to the satellite's CG.

then

$$\begin{aligned} \bar{Z}_{GPS} &= -\bar{R} \\ \bar{Y}_{GPS} &= \bar{Z}_{GPS} \times \bar{SS} \\ \bar{X}_{GPS} &= \bar{Y}_{GPS} \times \bar{Z}_{GPS} \end{aligned} \tag{8.9 -6}$$

GPS X axis acceleration

$$\hat{u} = \frac{\bar{X}_{GPS}}{|\bar{X}_{GPS}|} \tag{8.9 -7}$$

GPS Y axis acceleration

$$\hat{u} = \frac{\bar{Y}_{GPS}}{|\bar{Y}_{GPS}|} \tag{8.9 -8}$$

GPS Z axis acceleration

$$\hat{u} = \frac{\bar{Y}_{GPS}}{|\bar{Y}_{GPS}|} \tag{8.9 -9}$$

The variational partials are defined simply as

$$\bar{f} = \bar{u} \tag{8.9 -10}$$

8.10 DYNAMIC POLAR MOTION

The non-rigidity of the Earth is manifested in the temporal variability of its moments of inertia in response to both rotational and tidal deformations. The Earth's axis of figure, which is the principal axis of angular momentum, exhibits two periodic motions. A daily motion due to the Earth's response to the tidal deformation and a much smaller motion, with a period similar to that of the Chandlerian wobble which is the Earth's response to the rotational deformation. The latter is also known as dynamic polar motion.

This motion has been shown to be proportional to the main wobble with a proportionality factor of $K = \frac{k}{k_s}$, where:

k is the Earth's Love number (0.30)

k_s is the so-called "Secular Love number" (0.94), when the geopotential is referenced on the center of that wobble.

It is well known that the orientation of the axis of figure with respect to some arbitrary frame of reference is reflected in the values of the second degree, order one, harmonic of the spherical harmonic expansion of the gravitational field of the body C(2,1), S(2,1). If the geopotential is referenced on the eTRS (Conventional Terrestrial Reference System). then one has to account for the offset of that reference pole from the current center of the wobble.

In GEODYN, we use a general formulation which accounts for the temporal variation of the figure axis through the application of the proportional variations of the C(2,1) and S(2,1) harmonics. If we denote the proportionality factor by K the resulting model is:

$$\begin{aligned} C_{2,1}(t) &= \hat{C}_{2,1}(t_0) + \dot{C}_{2,1}(t - t_0) + K x_p(t) \hat{C}_{2,0} \\ S_{2,1}(t) &= \hat{S}_{2,1}(t_0) + \dot{S}_{2,1}(t - t_0) + K y_p(t) \hat{S}_{2,0} \end{aligned} \quad (8.10\text{ -1})$$

where $C_{2,1}, S_{2,1}$ are the values of these harmonics relative to the CTRS at an initial epoch, t_0.

The periodic part, which is represented by the last term, will average out in each Chandler cycle.

The center of the polar motion migrates slowly, and after some time, accumulates as an offset. To the extent that this offset (first term) becomes much larger than that of the periodic part, the second term is included to compensate for this future secular motion. [8 - 32]

8.11 EARTH RADIATION

The evaluation of the acceleration on a satellite due to the Earth's radiation pressure, requires integrating the effect of the albedo and emissivity over the surface of the Earth. In the computation of the acceleration, the satellite visible area of the Earth is divided into 19 elements of equal attenuated projected area. The acceleration from each of the 19 surface elements is summed together to approximate the actual surface integral. In computing the albedo and emissivity GEODYN uses the Knocke second degree zonal spherical harmonic representation of the Earth's albedo and emissivity. The earth radiation acceleration, due to a single earth area element, is given by [8-24]

$$d\vec{A}_{ER} = C_R(aE_S \cos\Theta_S + \frac{eE_s}{4})\frac{\cos\alpha A_S}{mC\pi\Gamma^2} dA\hat{\Gamma} \quad (8.11\text{ -1})$$

where

a = albedo

C_R = 1 + ρ_s (where ρ_s is the satellite reflectivity)

e = emissivity

E_S = solar irradiance (watts m^{-2})

Γ = distance from earth element to satellite

$\hat{\Gamma}$ = unit vector pointing from earth element to satellite

C = speed of light

A_S = cross sectional area of satellite

m = mass of satellite

α = view angle (angle between earth area element normal and earth area element-satellite vector)

dA = area of earth element

Using a summation of earth area elements (visible to the satellite), to approximate the integral, the earth radiation acceleration is given by

$$\vec{A}_{ER} = \sum_{J=1}^{N} C_R(K_{D_j} a_j E_S \cos\Theta_{S_j} + \frac{e_j E_s}{4}) \frac{A_S}{mC\pi\Gamma^2} \cos\alpha_j dA_j \hat{\Gamma}_j \tag{8.11 -2}$$

where

N is the total number of earth area elements

K_D is equal to 0 when the earth element is in darkness and 1 when the earth element is in sunlight

In the computation of the acceleration the visible area of the earth is divided into 19 elements, such that the attenuated projected areas for all of the elements are equal. The attenuated projected area is given by [8-24]

$$A' = \frac{dA_j \cos\alpha_j}{\pi\Gamma_j^2} \tag{8.11 -3}$$

A spherical harmonic expansion is used to calculate the albedo and emissivity for each of the earth area elements [8-24]. Therefore, the albedo and emissivity are given by

$$\begin{aligned} a &= a_0 + a_1 P_1(\sin\phi) + a_2 P_2(\sin\phi) \\ e &= e_0 + e_1 P_1(\sin\phi) + e_2 P_2(\sin\phi) \\ a_1 &= c_0 + c_1 \cos(\omega(JD - t_0)) + c_2 \sin(\omega(JD - t_0)) \\ e_1 &= K_0 + K_1 \cos(\omega(JD - t_0)) + K_2 \sin(\omega(JD - t_0)) \end{aligned} \tag{8.11 -4}$$

where

t_0 = epoch of periodic terms (December 22, 1981)

ω = $\frac{2\pi}{365.25}$days

a_0 = 0.34

a_2 = 0.29

e_0 = 0.68

e_2 = -0.18

c_0 = 0

c_1 = 0.10

c_2 = 0

K_0 = 0

K_1 = -0.07

K_2 = 0

8.12 THERMAL DRAG EFFECT ON LAGEOS

The drag on LAGEOS has been attributed to geophysical effects and satellite-specific effects [8 - 33]. An example of the second type of effect is the Yarkovsky thermal drag, which is due to LAGEOS's spin and thermal properties. Rubincam (8 - 28) and (8 - 33) has shown that the drag force lies along the spin axis and can account for 42 - 75% of the observed along-track acceleration, depending on the spin axis position. The remainder is accounted for by charged and neutral particle drag.

The thermal drag acceleration on LAGEOS (or LAGEOS type satellites), $\vec{a}_{TD}$ depends on the position of it's spin axis [8 - 27] and [8 - 28]. It is computed from:

$$\vec{a}_{TD} = \left[\frac{-D(\vec{r}_d \cdot \vec{S})}{(1+\zeta^2)^{\frac{1}{2}}} \right] \vec{S} \tag{8.12 -1}$$

Where D is a model constant, ζ a parameter, $\vec{S}$ the unit vector in the direction of the LAGEOS spin axis and $\vec{r}_d$ the unit vector from the geocenter to a previous (earlier) position of the satellite. The recommended parameter values which are the default values in GEODYN II are the following:

$D = -11.836 \times 10^{-12} ms^{-2}$

$\zeta = 1.2$

$\vec{S} = \pm(\sin\theta\cos\lambda\vec{x}, \, \sin\theta\sin\lambda\vec{y}, \, \cos\theta\vec{z})$, $\theta = 22°$ and $\lambda = 313°$ where θ is the colatitude of the satellite's spin axis and λ the right ascension.

The unit vector $\vec{r}_d$ is defined as the satellite's position vector obtained by going backwards in time from the current position vector by a time corresponding to the orbit angle $\delta = \arctan\zeta$.

The computation of $\vec{r}_d$ is done approximately using the osculating Keplerian elements.

$$\vec{r}_d \cong \begin{bmatrix} [\cos\Omega\cos(\omega + M - \delta) - \cos i \sin\Omega\sin(\omega + M - \delta)]\vec{x} \\ [\sin\Omega\cos(\omega + M - \delta) - \cos i \cos\Omega\sin(\omega + M - \delta)]\vec{y} \\ [\sin i \sin\Omega\sin(\omega + M - \delta)]\vec{z} \end{bmatrix} \tag{8.12 -2}$$

In the implementation of eq. (8.12-2) in GEODYN II, $\vec{a}_{TD}$ is calculated in the earth centered true of date at the integration epoch coordinate system. In addition, the Keplerian elements are rigorously computed from the satellite state vector every 12 hours, while for every integration step within this time span, only the mean anomaly (M) is updated from

$$M = M_0 + \eta\Delta t \tag{8.12 -3}$$

where η is the satellite's mean motion, M_0 is the rigorously computed mean anomaly at time t_0 and $\Delta t = t - t_0$.

To invoke this model for any arc, the option card THRDRG must be used with the satellite's ID specified. Also, the default values for D, δ and θ, λ, may be overridden by specifying new values on the THRDRG card.

8.13 RELATIVISTIC CORIOLIS FORCE

The relativistic Coriolis force (also known as geodetic or geodesic precession) represents the precession of the axis of a freely falling inertial frame. Thus, the inertial planet-centered frame precesses with respect to the inertial barycentric frame with angular velocity (if the sun is considered the only contributor given by [8 - 29] :

$$\vec{\Omega} = \frac{3}{2}(\vec{V}_E - \vec{V}_S) \times \left[\frac{-GM_S\vec{X}_{ES}}{c^2 R_{ES}^3}\right] \tag{8.13 -1}$$

where:

$\vec{V}_E$ = barycentric velocity of the central body

$\vec{V}_S$ = barycentric velocity of the sun

$\vec{X}_{ES} = \vec{X}_E - \vec{X}_S$

$R_{ES} = |\vec{X}_E - \vec{X}_S|$

c = speed of light

This precession exhibits itself as acceleration on a space-craft orbiting the central body much like the Coriolis force. Therefore, the contribution to the equations of motion implemented in Geodyn is given by:

$$\vec{a} = (\vec{\Omega} \times \vec{v}) \tag{8.13 -2}$$

where:

$\vec{v}$ = inertial planet-centered velocity of the space-craft

8.14 LENSE-THIRRlNG

Lense-Thirring acceleration is due to the mass current and gravito-magnetic field of a rotating gravitating body. A very simple explanation for this is given in Soffel, 1989; "A rotating central body influences the surrounding space-time in some sense similar as if it were immersed in a viscous fluid transferring some of its rotational energy to the surrounding medium." [8 - 30]. The effect on an orbiting space-craft is to drag the angular momentum vector of the orbit along with the rotating central body. The contribution to the equations of motion is implemented in Geodyn as

$$\begin{aligned} a_x &= \frac{4}{5}\frac{GM\omega R^2}{c^2 r^3}\left[\left(\frac{x^2+y^2-2z^2}{r^2}\right)\dot{y} + 3\left(\frac{yz}{r^2}\right)\dot{z}\right] \\ a_y &= \frac{4}{5}\frac{GM\omega R^2}{c^2 r^3}\left[-\left(\frac{x^2+y^2-2z^2}{r^2}\right)\dot{x} - 3\left(\frac{xz}{r^2}\right)\dot{z}\right] \\ a_z &= \frac{4}{5}\frac{GM\omega R^2}{c^2 r^3}\left[3\left(\frac{z}{r}\right)\left(\frac{x\dot{y}-y\dot{x}}{r}\right)\right] \end{aligned} \tag{8.14 -1}$$

where:

ω = angular velocity of the central body

R = radius of the central body

$r = \sqrt{x^2+y^2+z^2}$

8.15 PLANETARY MOON POINT MASS PERTURBATION

For central bodies other than the Earth, the Earth's Moon, and the Sun planetary moon point mass perturbations for up to 5 moons can be requested (see the PLMOON card in VOLUME 3). The contribution to the equations of motion for a particular moon is implemented in GEODYN in the following manner:

DIRECT EFFECT (direct gravitational force on the space-craft (s/c) from the moon):

$$\vec{a} = -\frac{GM\vec{R}}{R^3}; \vec{R} = \vec{R}_{s/c} - \vec{R}_{MOON}$$

$\vec{R}_{s/c}$ = Central body centered position vector of space-craft.

$\vec{R}_{MOON}$ = Central body centered position vector of the moon.

GM = The product of the Universal gravitational constant and the moon's mass.

INDIRECT EFFECT (The moon perturbs the origin of the coordinate system (the central body). Thus, the acceleration on the origin translates to an acceleration on the s/c):

$$\vec{a} = -\frac{GM\vec{R}_{MOON}}{R^3_{MOON}}; \tag{8.15 -1}$$

$\vec{R}_{MOON}$ = Central body centered position vector of the moon.

GM = The product of the Universal gravitational constant and the moon's mass.

The V-matrix partials are implemented in GEODYN in the following manner (partials are of the sum of both the direct and indirect accelerations):

V-MATRIX PARTIALS:

$$\frac{\partial a_x}{\partial x_{s/c}} = -\frac{GM}{R^3} + \frac{3GM(x_{s/c} - x_{MOON})^2}{R^5}; \vec{R} = \vec{R}_{s/c} - \vec{R}_{MOON}$$

$$\frac{\partial a_x}{\partial y_{s/c}} = \frac{3GM(x_{s/c} - x_{MOON})(y_{s/c} - y_{MOON})}{R^5}$$

$$\frac{\partial a_x}{\partial z_{s/c}} = \frac{3GM(x_{s/c} - x_{MOON})(z_{s/c} - z_{MOON})}{R^5}$$

*Partials of the y and z components of the acceleration are done in a similar manner.

The explicit variational equation partials for a particular moon's GM are implemented in GEODYN in the following manner (partials are of the sum of both the direct and indirect accelerations):

EXPLICIT PARTIALS:

$$\frac{\partial \vec{a}}{\partial (GM)} = -\left[\frac{\vec{R}}{R^3} + \frac{\vec{R}_{MOON}}{R^3_{MOON}} \right]$$
$$\vec{R} = \vec{R}_{s/c} - \vec{R}_{MOON}$$

8.16 ORIENTATION INITIAL CONDITIONS - VARIATIONAL EQUATIONS: T. Sabaka

8.16.1 Introduction

Let the position and velocity vectors at time t be given by $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, respectively, and let

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \\ \dot{\mathbf{r}}(t) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \dot{\mathbf{r}}(t) \\ \\ \ddot{\mathbf{r}}(t) \end{pmatrix}, \qquad (8.16\text{ -1})$$

where $\mathbf{f}(t)$ represents a generalized force model on $\mathbf{x}(t)$. It follows that

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{p}) \\ \\ \mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\tau, \mathbf{x}(\tau), \mathbf{p})\, d\tau \end{cases}, \qquad (8.16\text{ -2})$$

where $\mathbf{x}_0 = \mathbf{x}(0)$ and $\mathbf{p}$ represents a vector of parameters that influence the force model.

We obtain partial derivatives of $\mathbf{x}(t)$ with respect to the initial conditions $\mathbf{x}_0$ as follows

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t) = \mathbf{I} + \int_0^t \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau) \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\tau) d\tau \\ \\ \mathbf{\Phi}(t) = \mathbf{\Phi}(0) + \int_0^t \mathbf{F}(\tau) \mathbf{\Phi}(\tau) d\tau \\ \\ \dot{\mathbf{\Phi}}(t) = \mathbf{F}(t)\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I} \end{cases}, \qquad (8.16\text{ -3})$$

where $\mathbf{\Phi}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t)$ and $\mathbf{F}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)$.

Likewise, we obtain partial derivatives of $\mathbf{x}(t)$ with respect to the parameters $\mathbf{p}$ as follows

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t) &= \int_0^t \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau)\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\tau) + \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(\tau)\right] d\tau \\ \mathbf{\Psi}(t) &= \int_0^t \left[\mathbf{F}(\tau)\mathbf{\Psi}(\tau) + \mathbf{G}(\tau)\right] d\tau \\ \dot{\mathbf{\Psi}}(t) &= \mathbf{F}(t)\mathbf{\Psi}(t) + \mathbf{G}(t), \quad \mathbf{\Psi}(0) = \mathbf{0} \end{cases}, \tag{8.16 -4}$$

where $\mathbf{\Psi}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t)$ and $\mathbf{G}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(t)$.

8.16.2 Application to computing angular velocity and reference frame through time

Let the angular momentum of a rotating body in the inertial frame, denoted by "∼", be given by

$$\tilde{\mathbf{L}}(t) = \tilde{\mathbf{I}}_c\tilde{\boldsymbol{\omega}}(t), \tag{8.16 -5}$$

where $\tilde{\mathbf{I}}_c$ is the moment of inertia tensor and $\tilde{\boldsymbol{\omega}}$ is the angular velocity. If $\mathbf{R}(t)$ is an orthogonal matrix whose columns represent the principal axes of the body in the inertial frame, then

$$\begin{aligned} \dot{\mathbf{R}}(t) &= \dot{\mathbf{R}}(t)\mathbf{R}^{\mathrm{T}}(t)\mathbf{R}(t), \\ &= [\mathbf{R}(t)\boldsymbol{\omega}(t)] \times \mathbf{R}(t), \\ &= \tilde{\boldsymbol{\omega}}(t) \times \mathbf{R}(t), \\ &= \mathbf{f}_{\mathbf{R}}(t, \boldsymbol{\omega}(t), \mathbf{R}(t)). \end{aligned} \tag{8.16 -6}$$

The torque $\tilde{\mathbf{T}}(t)$ is given by

$$\begin{aligned} \tilde{\mathbf{T}}(t) &= \dot{\tilde{\mathbf{L}}}(t), \\ &= \frac{\partial}{\partial t}\left[\mathbf{R}(t)\mathbf{I}_c\boldsymbol{\omega}(t)\right], \\ &= \dot{\mathbf{R}}(t)\mathbf{I}_c\boldsymbol{\omega}(t) + \mathbf{R}(t)\mathbf{I}_c\dot{\boldsymbol{\omega}}(t), \\ &= [\mathbf{R}(t)\boldsymbol{\omega}(t)] \times [\mathbf{R}(t)\mathbf{I}_c\boldsymbol{\omega}(t)] + \mathbf{R}(t)\mathbf{I}_c\dot{\boldsymbol{\omega}}(t), \\ \mathbf{R}(t)\mathbf{T}(t) &= \mathbf{R}(t)\left[\mathbf{I}_c\dot{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}(t) \times \mathbf{I}_c\boldsymbol{\omega}(t)\right], \\ \mathbf{T}(t) &= \mathbf{I}_c\dot{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}(t) \times \mathbf{I}_c\boldsymbol{\omega}(t), \end{aligned} \tag{8.16 -7}$$

where the second term describes the time rate of change of the rotating reference frame defined by $\mathbf{R}(t)$. Therefore, all quantities in eq. 8.16 -7 are in the body-fixed frame. This leads to

$$\begin{aligned}\dot{\boldsymbol{\omega}}(t) &= \mathbf{I}_c^{-1}\left[\mathbf{T}(t) - \boldsymbol{\omega}(t)\times\mathbf{I}_c\boldsymbol{\omega}(t)\right],\\ &= \mathbf{I}_c^{-1}\left[\mathbf{R}^{\mathrm{T}}(t)\tilde{\mathbf{T}}(t) - \boldsymbol{\omega}(t)\times\mathbf{I}_c\boldsymbol{\omega}(t)\right],\\ &= \mathbf{f}_{\boldsymbol{\omega}}(t,\boldsymbol{\omega}(t),\mathbf{R}(t)).\end{aligned} \tag{8.16 -8}$$

Physically, the torque $\tilde{\mathbf{T}}(t)$ is produced by the non-central force of the body's gravitational field on, say, the sun, whose position vector $\tilde{\mathbf{r}}$ is given with respect to the body's center of mass. The gravity field acceleration is given by

$$\tilde{\mathbf{g}}(t) = \mathbf{R}(t)\mathbf{R}_{BS}(t,\theta,\phi)\bar{\mathbf{g}}(t,r,\theta,\phi), \tag{8.16 -9}$$

where (r,θ,ϕ) are the spherical coordinates of the sun's position vector in the body-fixed frame given by $\mathbf{r}(t) = \mathbf{R}^{\mathrm{T}}(t)\tilde{\mathbf{r}}$, $\bar{\mathbf{g}}(t,r,\theta,\phi)$ is the gravity acceleration on the sun in the local spherical frame due to the body's gravitational field, and $\mathbf{R}_{BS}(t,\theta,\phi)$ is the rotation matrix from the local spherical to the body-fixed frame. The equal and opposite force exerted on the body due to the sun is

$$\tilde{\mathbf{f}}_{sun}(t) = -M_{sun}\tilde{\mathbf{g}}(t), \tag{8.16 -10}$$

where M_{sun} is the solar mass, which leads to

$$\tilde{\mathbf{T}}(t) = \tilde{\mathbf{r}}\times\tilde{\mathbf{f}}_{sun}(t). \tag{8.16 -11}$$

Now, we are interested in the time evolution of both $\boldsymbol{\omega}(t)$ and $\mathbf{R}(t)$ and eqs. 8.16 -6 and 8.16 -8 allow us to use the procedure analogous to eq. 8.16 -2 such that

$$\left\{\begin{aligned}\dot{\boldsymbol{\omega}}(t) &= \mathbf{f}_{\boldsymbol{\omega}}(t,\boldsymbol{\omega}(t),\mathbf{R}(t))\\ \boldsymbol{\omega}(t) &= \boldsymbol{\omega}_0 + \textstyle\int_0^t \mathbf{f}_{\boldsymbol{\omega}}(\tau,\boldsymbol{\omega}(\tau),\mathbf{R}(\tau))\,d\tau\end{aligned}\right., \tag{8.16 -12}$$

where $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$, and

$$\left\{\begin{aligned}\dot{\mathbf{R}}(t) &= \mathbf{f}_{\mathbf{R}}(t,\boldsymbol{\omega}(t),\mathbf{R}(t))\\ \mathbf{R}(t) &= \mathbf{R}_0 + \textstyle\int_0^t \mathbf{f}_{\mathbf{R}}(\tau,\boldsymbol{\omega}(\tau),\mathbf{R}(\tau))\,d\tau\end{aligned}\right., \tag{8.16 -13}$$

where $\mathbf{R}_0 = \mathbf{R}(0)$.

8.16.3 Application to computing partials of angular velocity and reference frame with respect to initial conditions through time

We now wish to obtain partial derivatives of $\boldsymbol{\omega}(t)$ and $\mathbf{R}(t)$ with respect to the initial conditions $\boldsymbol{\omega}_0$ and $\mathbf{R}_0$ analogously to eq. 8.16 -3. However, we first introduce the Kronecker product of two matrices, $\mathbf{A}$ and $\mathbf{B}$, defined to be

$$\mathbf{A} \otimes \mathbf{B} \equiv \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}, \tag{8.16 -14}$$

where $\mathbf{A} = (a_{ij})$ is an $m \times n$ matrix. If "$vec(\cdot)$" is a matrix to vector operator that sequentially stacks the columns of a matrix on top of one another, then the following useful properties can be derived

$$vec\,(\mathbf{AXB}) = \left(\mathbf{B}^{\mathrm{T}} \otimes \mathbf{A}\right) vec\,(\mathbf{X})\,, \tag{8.16 -15}$$

$$\frac{\partial\,(vec\,(\mathbf{AXB}))}{\partial\,(vec\,(\mathbf{X}))} = \mathbf{B}^{\mathrm{T}} \otimes \mathbf{A}, \tag{8.16 -16}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{X}$ are three compatible matrices under the indicated multiplication. Further properties that will be useful are

$$vec\,(\mathbf{u}) = vec\left(\mathbf{u}^{\mathrm{T}}\right), \tag{8.16 -17}$$

$$\mathbf{u}\times = \mathbf{E}_{\mathbf{u}}, \tag{8.16 -18}$$

$$(\mathbf{Ru})\,\times = \mathbf{R}\mathbf{E}_{\mathbf{u}}\mathbf{R}^{\mathrm{T}}, \tag{8.16 -19}$$

where

$$\mathbf{E}_{\mathbf{u}} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}. \tag{8.16 -20}$$

The key is to compute the partial derivatives $\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t)$, $\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}}(t)$, $\frac{\partial \dot{\mathbf{R}}}{\partial \boldsymbol{\omega}}(t)$, and $\frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{R}}(t)$ where we now treat $\mathbf{R}$ as a vector quantity, i.e., $vec\,(\mathbf{R})$.

8.16.3.1 Computing $\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t)$ Here we rewrite eq. 8.16 -8 in order to differentiate with respect to the two factors of $\boldsymbol{\omega}(t)$, which we denote $\boldsymbol{\omega}_1(t)$ and $\boldsymbol{\omega}_2(t)$. The partial with respect to $\boldsymbol{\omega}_1(t)$ is then

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}_1}(t) &= \frac{\partial}{\partial \boldsymbol{\omega}_1}\left\{\mathbf{I}_c^{-1}\left[\mathbf{I}_c \boldsymbol{\omega}_2(t) \times \boldsymbol{\omega}_1(t)\right]\right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}_1}\left\{\mathbf{I}_c^{-1}\mathbf{E}_{\mathbf{I}_c\boldsymbol{\omega}_2}(t)\boldsymbol{\omega}_1(t)\right\}, \\
&= \mathbf{I}_c^{-1}\mathbf{E}_{\mathbf{I}_c\boldsymbol{\omega}_2}(t), \qquad (8.16\text{ -21})
\end{aligned}$$

where $\mathbf{E}_{\mathbf{I}_c\boldsymbol{\omega}_2}(t) = \mathbf{I}_c\boldsymbol{\omega}_2(t)\times$, and the partial with respect to $\boldsymbol{\omega}_2(t)$ is

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}_2}(t) &= \frac{\partial}{\partial \boldsymbol{\omega}_2}\left\{-\mathbf{I}_c^{-1}\left[\boldsymbol{\omega}_1(t) \times \mathbf{I}_c\boldsymbol{\omega}_2(t)\right]\right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}_2}\left\{-\mathbf{I}_c^{-1}\mathbf{E}_{\boldsymbol{\omega}_1}(t)\mathbf{I}_c\boldsymbol{\omega}_2(t)\right\}, \\
&= -\mathbf{I}_c^{-1}\mathbf{E}_{\boldsymbol{\omega}_1}(t)\mathbf{I}_c. \qquad (8.16\text{ -22})
\end{aligned}$$

Combining eqs. 8.16 -21 and 8.16 -22 gives the final result

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t) &= \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}_1}(t) + \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}_2}(t), \\
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t) &= \mathbf{I}_c^{-1}\left[\mathbf{E}_{\mathbf{I}_c\boldsymbol{\omega}}(t) - \mathbf{E}_{\boldsymbol{\omega}}(t)\mathbf{I}_c\right]. \qquad (8.16\text{ -23})
\end{aligned}$$

8.16.3.2 Computing $\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}}(t)$ Again, we rewrite eq. 8.16 -8 to show dependence on two instances of $\mathbf{R}(t)$, indicated by $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$. Taking the partial with respect to $\mathbf{R}_1(t)$ gives

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}_1}(t) &= \frac{\partial}{\partial \mathbf{R}_1}\left\{\mathbf{I}_c^{-1}\mathbf{R}_1^{\mathrm{T}}(t)\tilde{\mathbf{T}}(t, \mathbf{R}_2(t))\right\}, \\
&= \frac{\partial}{\partial \mathbf{R}_1}\left\{\mathbf{I}_c^{-1} vec\left[\tilde{\mathbf{T}}^{\mathrm{T}}(t, \mathbf{R}_2(t))\mathbf{R}_1(t)\right]\right\}, \\
&= \frac{\partial}{\partial \mathbf{R}_1}\left\{\mathbf{I}_c^{-1} vec\left[\tilde{\mathbf{T}}^{\mathrm{T}}(t, \mathbf{R}_2(t))\mathbf{R}_1(t)\mathbf{I}\right]\right\}, \\
&= \mathbf{I}_c^{-1}\left[\mathbf{I} \otimes \tilde{\mathbf{T}}^{\mathrm{T}}(t, \mathbf{R}_2(t))\right], \qquad (8.16\text{ -24})
\end{aligned}$$

where $\mathbf{I}$ is an appropriate identity matrix. Taking the partial with respect to $\mathbf{R}_2(t)$ gives

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}_2}(t) &= \frac{\partial}{\partial \mathbf{R}_2}\left\{\mathbf{I}_c^{-1}\mathbf{R}_1^{\mathrm{T}}(t)\tilde{\mathbf{T}}(t, \mathbf{R}_2(t))\right\}, \\
&= \mathbf{I}_c^{-1}\mathbf{R}_1^{\mathrm{T}}(t)\frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}_2}(t, \mathbf{R}_2(t)), \qquad (8.16\text{ -25})
\end{aligned}$$

where $\frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}}(t)$ is yet to be evaluated. Therefore,

$$\begin{aligned} \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}}(t) &= \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}_1}(t) + \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}_2}(t), \\ &= \mathbf{I}_c^{-1} \left[\mathbf{I} \otimes \tilde{\mathbf{T}}^{\mathrm{T}}(t, \mathbf{R}(t)) + \mathbf{R}^{\mathrm{T}}(t) \frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}}(t, \mathbf{R}(t)) \right]. \end{aligned} \tag{8.16 -26}$$

We now focus on the evaluation of $\frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}}(t)$ from eq. 8.16 -11

$$\begin{aligned} \frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}}(t) &= \frac{\partial}{\partial \mathbf{R}} \left\{ \tilde{\mathbf{r}} \times \tilde{\mathbf{f}}_{sun}(t, \mathbf{R}(t)) \right\}, \\ &= \frac{\partial}{\partial \mathbf{R}} \left\{ \mathbf{E}_{\tilde{\mathbf{r}}} \tilde{\mathbf{f}}_{sun}(t, \mathbf{R}(t)) \right\}, \\ &= \frac{\partial}{\partial \mathbf{R}} \left\{ \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}(t) \mathbf{R}_{BS}(t, \mathbf{R}(t)) \bar{\mathbf{f}}_{sun}(t, \mathbf{R}(t)) \right\}. \end{aligned} \tag{8.16 -27}$$

We now break eq. 8.16 -27 into two instances of $\mathbf{R}(t)$, denoted $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$. Taking the partial with respect to $\mathbf{R}_1(t)$ gives

$$\begin{aligned} \frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}_1}(t) &= \frac{\partial}{\partial \mathbf{R}_1} \left\{ \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}_1(t) \mathbf{R}_{BS}(t, \mathbf{R}_2(t)) \bar{\mathbf{f}}_{sun}(t, \mathbf{R}_2(t)) \right\}, \\ &= \frac{\partial}{\partial \mathbf{R}_1} \left\{ \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}_1(t) \mathbf{f}_{sun}(t, \mathbf{R}_2(t)) \right\}, \\ &= \mathbf{f}_{sun}^{\mathrm{T}}(t, \mathbf{R}_2(t)) \otimes \mathbf{E}_{\tilde{\mathbf{r}}}, \end{aligned} \tag{8.16 -28}$$

where $\mathbf{f}_{sun}(t, \mathbf{R}_2(t))$ is the force in the body-fixed frame. Taking the partial with respect to $\mathbf{R}_2(t)$ gives

$$\begin{aligned} \frac{\partial \tilde{\mathbf{T}}}{\partial \mathbf{R}_2}(t) &= \frac{\partial}{\partial \mathbf{R}_2} \left\{ \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}_1(t) \mathbf{R}_{BS}(t, \mathbf{R}_2(t)) \bar{\mathbf{f}}_{sun}(t, \mathbf{R}_2(t)) \right\}, \\ &= \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}_1(t) \frac{\partial}{\partial \mathbf{R}_2} \left\{ \mathbf{R}_{BS}(t, \mathbf{R}_2(t)) \bar{\mathbf{f}}_{sun}(t, \mathbf{R}_2(t)) \right\}, \\ &= \mathbf{E}_{\tilde{\mathbf{r}}} \mathbf{R}_1(t) \frac{\partial}{\partial (r, \theta, \phi)} \left\{ \mathbf{R}_{BS}(t, \theta, \phi) \bar{\mathbf{f}}_{sun}(t, r, \theta, \phi) \right\} \frac{\partial (r, \theta, \phi)}{\partial \mathbf{r}}(t) \frac{\partial \mathbf{r}}{\partial \mathbf{R}_2}(t), \end{aligned} \tag{8.16 -29}$$

where, recall, (r, θ, ϕ) are the spherical coordinates of the sun's position vector $\mathbf{r}(t)$ in the body-fixed frame. Since

$$\mathbf{r}(t) = \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{8.16 -30}$$

then

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \tag{8.16 -31}$$

which lead to

$$\frac{\partial(r,\theta,\phi)}{\partial \mathbf{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}, \tag{8.16 -32}$$

and since $\mathbf{r}(t) = \mathbf{R}^{\mathrm{T}}(t)\tilde{\mathbf{r}}$, then

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \mathbf{R}}(t) &= \frac{\partial}{\partial \mathbf{R}}\left\{\mathbf{R}^{\mathrm{T}}(t)\tilde{\mathbf{r}}\right\}, \\ &= \frac{\partial}{\partial \mathbf{R}}\left\{vec\left[\tilde{\mathbf{r}}^{\mathrm{T}}\mathbf{R}(t)\right]\right\}, \\ &= \frac{\partial}{\partial \mathbf{R}}\left\{vec\left[\tilde{\mathbf{r}}^{\mathrm{T}}\mathbf{R}(t)\mathbf{I}\right]\right\}, \\ &= \mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}. \end{aligned} \tag{8.16 -33}$$

We now deal with partial derivatives of the rotation matrix $\mathbf{R}_{BS}(t,\theta,\phi)$ given by

$$\mathbf{R}_{BS}(t,\theta,\phi) = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}, \tag{8.16 -34}$$

which leads to

$$\frac{\partial \mathbf{R}_{BS}}{\partial r}(t) = \mathbf{0}, \tag{8.16 -35}$$

$$\begin{aligned} \frac{\partial \mathbf{R}_{BS}}{\partial \theta}(t) &= \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix}, \\ &= \mathbf{R}_{BR}(t,\theta,\phi)\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \mathbf{R}_{BR}(t,\theta,\phi)\mathbf{G}_{\theta}, \end{aligned} \tag{8.16 -36}$$

$$\begin{aligned} \frac{\partial \mathbf{R}_{BS}}{\partial \phi}(t) &= \begin{pmatrix} -\sin\theta\sin\phi & -\cos\theta\sin\phi & -\cos\phi \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \mathbf{R}_{BR}(t,\theta,\phi)\begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix}, \\ &= \mathbf{R}_{BR}(t,\theta,\phi)\mathbf{G}_{\phi}. \end{aligned} \tag{8.16 -37}$$

Therefore

$$\frac{\partial}{\partial(r,\theta,\phi)}\left\{\mathbf{R}_{BS}\bar{\mathbf{f}}_{sun}\right\} = \frac{\partial \mathbf{R}_{BS}}{\partial(r,\theta,\phi)}\bar{\mathbf{f}}_{sun}(t,r,\theta,\phi) + \mathbf{R}_{BS}(t,\theta,\phi)\frac{\partial \bar{\mathbf{f}}_{sun}}{\partial(r,\theta,\phi)},$$

$$= \mathbf{R}_{BS}(t,\theta,\phi)\left[\begin{array}{ccc} \frac{\partial \bar{\mathbf{f}}_{sun}}{\partial r} & \frac{\partial \bar{\mathbf{f}}_{sun}}{\partial \theta} + \mathbf{G}_{\theta}\bar{\mathbf{f}}_{sun} & \frac{\partial \bar{\mathbf{f}}_{sun}}{\partial \phi} + \mathbf{G}_{\phi}\bar{\mathbf{f}}_{sun} \end{array}\right] \quad (8.16\text{ -}38)$$

Finally, we deal with partial derivatives of $\bar{\mathbf{f}}_{sun}(t,r,\theta,\phi)$ with respect to (r,θ,ϕ). If the acceleration $\bar{\mathbf{g}}_{sun}(t,r,\theta,\phi) = \nabla U(t,r,\theta,\phi)$, where $U(t,r,\theta,\phi)$ is a potential function, then

$$\bar{\mathbf{g}}_{sun}(t,r,\theta,\phi) = \begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial U}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}(\ell+1)\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}\left[\cos(m\phi)C_{\ell}^{m} + \sin(m\phi)S_{\ell}^{m}\right]P_{\ell}^{m}(\theta) \\ \frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}\left[\cos(m\phi)C_{\ell}^{m} + \sin(m\phi)S_{\ell}^{m}\right]\frac{dP_{\ell}^{m}}{d\theta}(\theta) \\ \frac{GM}{r^2\sin\theta}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}m\left[\cos(m\phi)S_{\ell}^{m} - \sin(m\phi)C_{\ell}^{m}\right]P_{\ell}^{m}(\theta) \end{pmatrix}, \quad (8.16\text{ -}39)$$

where G is the gravitational constant, M is the mass of the body, a is the semi-major axis of the body, C_{ℓ}^{m} and S_{ℓ}^{m} are the Stokes coefficients of spherical harmonic degree ℓ and order m, $P_{\ell}^{m}(\theta)$ are the fully-normalized associated Legendre functions of degree ℓ and order m, and ℓ_{max} is the degree truncation level. It follows that

$$\frac{\partial \bar{\mathbf{g}}_{sun}}{\partial r} =$$

$$\begin{pmatrix} \frac{GM}{r^3}\sum_{\ell=0}^{\ell_{max}}(\ell+1)(\ell+2)\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}\left[\cos(m\phi)C_{\ell}^{m} + \sin(m\phi)S_{\ell}^{m}\right]P_{\ell}^{m}(\theta) \\ \frac{-GM}{r^3}\sum_{\ell=0}^{\ell_{max}}(\ell+2)\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}\left[\cos(m\phi)C_{\ell}^{m} + \sin(m\phi)S_{\ell}^{m}\right]\frac{dP_{\ell}^{m}}{d\theta}(\theta) \\ \frac{-GM}{r^3\sin\theta}\sum_{\ell=0}^{\ell_{max}}(\ell+2)\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}m\left[\cos(m\phi)S_{\ell}^{m} - \sin(m\phi)C_{\ell}^{m}\right]P_{\ell}^{m}(\theta) \end{pmatrix}, \quad (8.16\text{ -}40)$$

and

$$\frac{\partial \bar{\mathbf{g}}_{sun}}{\partial \theta} =$$

$$\begin{pmatrix} -\frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}(\ell+1)\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}\left[\cos(m\phi)C_\ell^m+\sin(m\phi)S_\ell^m\right]\frac{dP_\ell^m}{d\theta}(\theta) \\ \frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}\left[\cos(m\phi)C_\ell^m+\sin(m\phi)S_\ell^m\right]\frac{d^2P_\ell^m}{d\theta^2}(\theta) \\ \frac{GM}{r^2\sin\theta}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}m\left[\cos(m\phi)S_\ell^m-\sin(m\phi)C_\ell^m\right]\left[\frac{dP_\ell^m}{d\theta}(\theta)-\cot\theta P_\ell^m(\theta)\right] \end{pmatrix}, \quad (8.16\text{ -41})$$

and

$$\frac{\partial\bar{\mathbf{g}}_{sun}}{\partial\phi} =$$

$$\begin{pmatrix} -\frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}(\ell+1)\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}m\left[\cos(m\phi)S_\ell^m-\sin(m\phi)C_\ell^m\right]P_\ell^m(\theta) \\ \frac{GM}{r^2}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}m\left[\cos(m\phi)S_\ell^m-\sin(m\phi)C_\ell^m\right]\frac{dP_\ell^m}{d\theta}(\theta) \\ -\frac{GM}{r^2\sin\theta}\sum_{\ell=0}^{\ell_{max}}\left(\frac{a}{r}\right)^\ell\sum_{m=0}^{\ell}m^2\left[\cos(m\phi)C_\ell^m+\sin(m\phi)S_\ell^m\right]P_\ell^m(\theta) \end{pmatrix}. \quad (8.16\text{ -42})$$

We use the associated Legendre differential equation to compute $\frac{d^2P_\ell^m}{d\theta^2}(\theta)$ as

$$\frac{d^2P_\ell^m}{d\theta^2}(\theta) = \left[\frac{m^2}{\sin^2\theta}-\ell(\ell+1)\right]P_\ell^m(\theta)-\cot\theta\frac{dP_\ell^m}{d\theta}(\theta). \quad (8.16\text{ -43})$$

The pieces are now assembled to evaluate eq. 8.16 -26 using

$$\begin{aligned} \frac{\partial\tilde{\mathbf{T}}}{\partial\mathbf{R}}(t) &= \frac{\partial\tilde{\mathbf{T}}}{\partial\mathbf{R}_1}(t)+\frac{\partial\tilde{\mathbf{T}}}{\partial\mathbf{R}_2}(t), \\ &= \left(\mathbf{f}_{sun}^{\mathrm{T}}\otimes\mathbf{E}_{\tilde{\mathbf{r}}}\right)+\mathbf{E}_{\tilde{\mathbf{r}}}\mathbf{R}\frac{\partial}{\partial(r,\theta,\phi)}\left\{\mathbf{R}_{BS}\bar{\mathbf{f}}_{sun}\right\}\frac{\partial(r,\theta,\phi)}{\partial\mathbf{r}}\left(\mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}\right), \\ &= \left(\mathbf{f}_{sun}^{\mathrm{T}}\otimes\mathbf{E}_{\tilde{\mathbf{r}}}\right)+\mathbf{E}_{\tilde{\mathbf{r}}}\mathbf{R}\mathbf{R}_{BS}\overline{\nabla}\bar{\mathbf{f}}_{sun}\mathbf{R}_{BS}^{\mathrm{T}}\left(\mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}\right), \\ &= \left(\mathbf{f}_{sun}^{\mathrm{T}}\otimes\mathbf{E}_{\tilde{\mathbf{r}}}\right)+\mathbf{E}_{\tilde{\mathbf{r}}}\mathbf{R}\nabla\mathbf{f}_{sun}\left(\mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}\right), \end{aligned} \quad (8.16\text{ -44})$$

where $\overline{\nabla}$ and ∇ are the gradient operators in the local spherical and body-fixed frames, respectively. We now write the final expression as

$$\frac{\partial\dot{\boldsymbol{\omega}}}{\partial\mathbf{R}}(t) = \mathbf{I}_c^{-1}\left\{\mathbf{I}\otimes\tilde{\mathbf{T}}^{\mathrm{T}}+\mathbf{R}^{\mathrm{T}}\left[\left(\mathbf{f}_{sun}^{\mathrm{T}}\otimes\mathbf{E}_{\tilde{\mathbf{r}}}\right)+\mathbf{E}_{\tilde{\mathbf{r}}}\mathbf{R}\nabla\mathbf{f}_{sun}\left(\mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}\right)\right]\right\}. \quad (8.16\text{ -45})$$

8.16.3.3 Computing $\frac{\partial \dot{\mathbf{R}}}{\partial \boldsymbol{\omega}}(t)$

We use eq. 8.16 -6 to evaluate this as

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{R}}}{\partial \boldsymbol{\omega}}(t) &= \frac{\partial}{\partial \boldsymbol{\omega}} \left\{ \tilde{\boldsymbol{\omega}}(t) \times \mathbf{R}(t) \right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}} \left\{ [\mathbf{R}(t)\boldsymbol{\omega}(t)] \times \mathbf{R}(t) \right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}} \left\{ \mathbf{R}(t)\mathbf{E}_{\boldsymbol{\omega}}(t)\mathbf{R}^{\mathrm{T}}(t)\mathbf{R}(t) \right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}} \left\{ \mathbf{R}(t)\mathbf{E}_{\boldsymbol{\omega}}(t) \right\}, \\
&= \frac{\partial}{\partial \boldsymbol{\omega}} \left\{ \mathbf{R}(t)\mathbf{E}_{\boldsymbol{\omega}}(t)\mathbf{I} \right\}, \\
&= [\mathbf{I} \otimes \mathbf{R}(t)] \frac{\partial \mathbf{E}_{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t), \\
&= [\mathbf{I} \otimes \mathbf{R}(t)] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0} & -\mathbf{R}_z(t) & \mathbf{R}_y(t) \\ \mathbf{R}_z(t) & \mathbf{0} & -\mathbf{R}_x(t) \\ -\mathbf{R}_y(t) & \mathbf{R}_x(t) & \mathbf{0} \end{pmatrix},
\end{aligned} \tag{8.16 -46}
$$

where $\mathbf{R}(t)$ is written in terms of its column vectors as

$$
\mathbf{R}(t) = \left[\begin{array}{ccc} \mathbf{R}_x(t) & \mathbf{R}_y(t) & \mathbf{R}_z(t) \end{array} \right]. \tag{8.16 -47}
$$

8.16.3.4 Computing $\frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{R}}(t)$

From the previous subsection we have

$$
\begin{aligned}
\frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{R}}(t) &= \frac{\partial}{\partial \mathbf{R}} \left\{ \mathbf{R}(t)\mathbf{E}_{\boldsymbol{\omega}}(t) \right\}, \\
&= \frac{\partial}{\partial \mathbf{R}} \left\{ \mathbf{I}\mathbf{R}(t)\mathbf{E}_{\boldsymbol{\omega}}(t) \right\}, \\
&= \mathbf{E}_{\boldsymbol{\omega}}^{\mathrm{T}}(t) \otimes \mathbf{I}, \\
&= -\mathbf{E}_{\boldsymbol{\omega}}(t) \otimes \mathbf{I}.
\end{aligned} \tag{8.16 -48}
$$

8.16.3.5 Variational equations We now wish to compute the partial derivatives of angular velocity and the reference frame with respect to their initial conditions analogous to eq. 8.16 -3, and so

$$\begin{cases} \boldsymbol{\Omega}(t) &= \boldsymbol{\Omega}(0) + \int_0^t \mathbf{F}(\tau)\boldsymbol{\Omega}(\tau)d\tau \\ \dot{\boldsymbol{\Omega}}(t) &= \mathbf{F}(t)\boldsymbol{\Omega}(t), \quad \boldsymbol{\Omega}(0) = \mathbf{I} \end{cases}, \tag{8.16 -49}$$

where

$$\boldsymbol{\Omega}(t) = \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\omega}_0}(t) & \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{R}_0}(t) \\ \frac{\partial \mathbf{R}}{\partial \boldsymbol{\omega}_0}(t) & \frac{\partial \mathbf{R}}{\partial \mathbf{R}_0}(t) \end{pmatrix}, \tag{8.16 -50}$$

and

$$\mathbf{F}(t) = \begin{pmatrix} \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}(t) & \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{R}}(t) \\ \frac{\partial \dot{\mathbf{R}}}{\partial \boldsymbol{\omega}}(t) & \frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{R}}(t) \end{pmatrix} =$$

$$\left(\begin{array}{c|c} \mathbf{I}_c^{-1}\left[\mathbf{E}_{\mathbf{I}_c\boldsymbol{\omega}} - \mathbf{E}_{\boldsymbol{\omega}}\mathbf{I}_c\right] & \mathbf{I}_c^{-1}\left\{\mathbf{I}\otimes\tilde{\mathbf{T}}^{\mathrm{T}} + \mathbf{R}^{\mathrm{T}}\left[\left(\mathbf{f}_{sun}^{\mathrm{T}}\otimes\mathbf{E}_{\tilde{\mathbf{r}}}\right) + \mathbf{E}_{\tilde{\mathbf{r}}}\mathbf{R}\nabla\mathbf{f}_{sun}\left(\mathbf{I}\otimes\tilde{\mathbf{r}}^{\mathrm{T}}\right)\right]\right\} \\ \hline \begin{array}{ccc} \mathbf{0} & -\mathbf{R}_z & \mathbf{R}_y \\ \mathbf{R}_z & \mathbf{0} & -\mathbf{R}_x \\ -\mathbf{R}_y & \mathbf{R}_x & \mathbf{0} \end{array} & -\mathbf{E}_{\boldsymbol{\omega}}\otimes\mathbf{I} \end{array}\right). \tag{8.16 -51}$$

8.16.4 Application to IAU reference frames

The IAU convention has determined the type of decomposition to be used for planetary orientation (rotation) matrices in eq. 8.16 -6. The rotation matrix transforming from J2000 to the planetary system (PS) is of the type

$$\mathbf{R}_{PS\leftarrow J2000} = \mathbf{R}_{\hat{\mathbf{z}}}(W)\mathbf{R}_{\hat{\mathbf{x}}}(\theta)\mathbf{R}_{\hat{\mathbf{z}}}(\phi), \tag{8.16 -52}$$

where $\theta = 90^\circ - DEC$ and $\phi = RA + 90^\circ$. The IAU angles RA and DEC describe the position of the planetary north pole as right-ascension and declination, and the angle W describes the position of the planetary prime meridian as an eastward rotation from the point of the x axis after rotations by RA and DEC.

We are interested in integrating the variational equations in $J2000$ and so we work with the inverse of this rotation matrix given by

$$\begin{aligned}
\mathbf{R}_{J2000\leftarrow PS}(t) &= \mathbf{R}_{\hat{\mathbf{z}}}(-\phi(t))\mathbf{R}_{\hat{\mathbf{x}}}(-\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(-W(t)), \\
&= \begin{pmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{pmatrix} \begin{pmatrix} C_W & -S_W & 0 \\ S_W & C_W & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&= \begin{pmatrix} C_\phi C_W - S_\phi C_\theta S_W & -C_\phi S_W - S_\phi C_\theta C_W & S_\phi S_\theta \\ S_\phi C_W + C_\phi C_\theta S_W & -S_\phi S_W + C_\phi C_\theta C_W & -C_\phi S_\theta \\ S_\theta S_W & S_\theta C_W & C_\theta \end{pmatrix}, \qquad (8.16\text{ -}53)
\end{aligned}$$

where $C_{[\cdot]}$ and $S_{[\cdot]}$ denote $\cos(\cdot)$ and $\sin(\cdot)$, respectively, of an angular argument. An expression for $\boldsymbol{\omega}(t)$ can be evaluated from eqs. 8.16 -6 and 8.16 -53 by noting that

$$\begin{aligned}
\mathbf{E}_{\boldsymbol{\omega}}(t) &= \mathbf{R}^{\mathrm{T}}(t)\dot{\mathbf{R}}(t), \\
&= \mathbf{R}_{\hat{\mathbf{z}}}(W(t))\mathbf{R}_{\hat{\mathbf{x}}}(\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(\phi(t)) \Big[\dot{\mathbf{R}}_{\hat{\mathbf{z}}}(-\phi(t))\mathbf{R}_{\hat{\mathbf{x}}}(-\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(-W(t)) + \\
&\quad \mathbf{R}_{\hat{\mathbf{z}}}(-\phi(t))\dot{\mathbf{R}}_{\hat{\mathbf{x}}}(-\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(-W(t)) + \mathbf{R}_{\hat{\mathbf{z}}}(-\phi(t))\mathbf{R}_{\hat{\mathbf{x}}}(-\theta(t))\dot{\mathbf{R}}_{\hat{\mathbf{z}}}(-W(t))\Big], \\
&= \begin{pmatrix} 0 & -C_\theta & S_\theta C_W \\ C_\theta & 0 & -S_\theta S_W \\ -S_\theta C_W & S_\theta S_W & 0 \end{pmatrix} \dot{\phi}(t) + \begin{pmatrix} 0 & 0 & -S_W \\ 0 & 0 & -C_W \\ S_W & C_W & 0 \end{pmatrix} \dot{\theta}(t) + \\
&\quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{W}(t), \qquad (8.16\text{ -}54)
\end{aligned}$$

which leads to

$$\begin{aligned}
\boldsymbol{\omega}(t) &= \begin{pmatrix} S_\theta S_W & C_W & 0 \\ S_\theta C_W & -S_W & 0 \\ C_\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \\ \dot{W}(t) \end{pmatrix}, \\
&= \mathbf{A}(t)\mathbf{a}(t), \qquad (8.16\text{ -}55)
\end{aligned}$$

where $\mathbf{A}(t)$ is a matrix whose columns represent the three rotation axes in the body-fixed system and $\mathbf{a}(t)$ is a vector of angular rates. It should be noted that eq. 8.16 -55 is in exact agreement with eqs. 10-12 of Rubincam's Binary Asteroid Notes where his $(\phi_A, \theta_A, \Psi_A)$ are equivalent to (ϕ, θ, W).

It should be clear that if given $RA(0)$, $DEC(0)$, $W(0)$, $\dot{RA}(0)$, $\dot{DEC}(0)$, and $\dot{W}(0)$, then eqs. 8.16 -53 and 8.16 -55 can be used to compute $\mathbf{R}_0 = \mathbf{R}(0)$ and $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$, respectively, given that $\theta(0) = 90^\circ - DEC(0)$, $\phi(0) = RA(0) + 90^\circ$, $\dot{\theta}(0) = -\dot{DEC}(0)$, and $\dot{\phi}(0) = \dot{RA}(0)$.

Now, after the estimation procedure we have updated values of the initial conditions $\mathbf{R}_0$

and $\boldsymbol{\omega}_0$ and so we would like to update the values of the IAU angles and their rates. From eq. 8.16 -53 we see that we can get $\phi(0)$, $\theta(0)$, and $W(0)$ from $\mathbf{R}_0$ as

$$\phi(0) = \tan^{-1}\left(\frac{\mathbf{R}_0(1,3)}{-\mathbf{R}_0(2,3)}\right), \quad \theta(0) = \cos^{-1}\mathbf{R}_0(3,3), \quad W(0) = \tan^{-1}\left(\frac{\mathbf{R}_0(3,1)}{\mathbf{R}_0(3,2)}\right), \tag{8.16 -56}$$

followed by $RA(0) = \phi(0) - 90°$ and $DEC(0) = 90° - \theta(0)$. From eqs. 8.16 -55 and 8.16 -56 we form $\mathbf{A}(0)$ and compute $\mathbf{a}(0) = \mathbf{A}^{-1}(0)\boldsymbol{\omega}(0)$, assuming $\mathbf{A}(0)$ is invertible, to get the rates $\dot{\phi}(0)$, $\dot{\theta}(0)$, and $\dot{W}(0)$. This leads to $\dot{RA}(0) = \dot{\phi}(0)$ and $D\dot{E}C(0) = -\dot{\theta}(0)$. We can now move freely between the initial IAU orientation angles and rates $RA(0)$, $DEC(0)$, $W(0)$, $\dot{RA}(0)$, $D\dot{E}C(0)$, and $\dot{W}(0)$ and the initial reference frame $\mathbf{R}_0$ and angular velocity $\boldsymbol{\omega}_0$.

8.16.4.1 Direct estimation of angles and their rates from observations that are functions of the angles

We have just discussed updating the IAU angles and their rates from estimates of $\boldsymbol{\omega}_0$ and $\mathbf{R}_0$, but we may wish to estimate these directly from observations, particularly observations that are functions of the angles. From eq. 8.16 -55 we compute

$$\frac{\partial \boldsymbol{\omega}}{\partial(\phi,\theta,W)}(t) = \left[\begin{array}{ccc} \frac{\partial \mathbf{A}}{\partial \phi}(t)\mathbf{a}(t) & \frac{\partial \mathbf{A}}{\partial \theta}(t)\mathbf{a}(t) & \frac{\partial \mathbf{A}}{\partial W}(t)\mathbf{a}(t) \end{array}\right], \tag{8.16 -57}$$

where

$$\frac{\partial \mathbf{A}}{\partial \phi}(t) = \mathbf{0}, \tag{8.16 -58}$$

$$\frac{\partial \mathbf{A}}{\partial \theta}(t) = \begin{pmatrix} C_\theta S_W & 0 & 0 \\ C_\theta C_W & 0 & 0 \\ -S_\theta & 0 & 0 \end{pmatrix}, \tag{8.16 -59}$$

$$\frac{\partial \mathbf{A}}{\partial W}(t) = \begin{pmatrix} S_\theta C_W & -S_W & 0 \\ -S_\theta S_W & -C_W & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.16 -60}$$

Also from eq. 8.16 -55 we get

$$\frac{\partial \boldsymbol{\omega}}{\partial(\dot{\phi},\dot{\theta},\dot{W})}(t) = \mathbf{A}(t). \tag{8.16 -61}$$

Now, from eq. 8.16 -53 we have

$$\frac{\partial \mathbf{R}}{\partial(\phi,\theta,W)}(t) = \left[\begin{array}{ccc} vec\left(\frac{\partial \mathbf{R}}{\partial \phi}(t)\right) & vec\left(\frac{\partial \mathbf{R}}{\partial \theta}(t)\right) & vec\left(\frac{\partial \mathbf{R}}{\partial W}(t)\right) \end{array}\right], \tag{8.16 -62}$$

where

$$\begin{aligned}
\frac{\partial \mathbf{R}}{\partial \phi}(t) &= \frac{\partial \mathbf{R}_{\hat{\mathbf{z}}}}{\partial \phi}(-\phi(t))\mathbf{R}_{\hat{\mathbf{x}}}(-\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(-W(t)), && (8.16\text{ -}63)\\
\frac{\partial \mathbf{R}}{\partial \theta}(t) &= \mathbf{R}_{\hat{\mathbf{z}}}(-\phi(t))\frac{\partial \mathbf{R}_{\hat{\mathbf{x}}}}{\partial \theta}(-\theta(t))\mathbf{R}_{\hat{\mathbf{z}}}(-W(t)), && (8.16\text{ -}64)\\
\frac{\partial \mathbf{R}}{\partial W}(t) &= \mathbf{R}_{\hat{\mathbf{z}}}(-\phi(t))\mathbf{R}_{\hat{\mathbf{x}}}(-\theta(t))\frac{\partial \mathbf{R}_{\hat{\mathbf{z}}}}{\partial W}(-W(t)), && (8.16\text{ -}65)
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial \mathbf{R}_{\hat{\mathbf{z}}}}{\partial \phi}(-\phi(t)) &= \begin{pmatrix} -S_\phi & -C_\phi & 0 \\ C_\phi & -S_\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}, && (8.16\text{ -}66)\\
\frac{\partial \mathbf{R}_{\hat{\mathbf{x}}}}{\partial \theta}(-\theta(t)) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -S_\theta & -C_\theta \\ 0 & C_\theta & -S_\theta \end{pmatrix}, && (8.16\text{ -}67)\\
\frac{\partial \mathbf{R}_{\hat{\mathbf{z}}}}{\partial W}(-W(t)) &= \begin{pmatrix} -S_W & -C_W & 0 \\ C_W & -S_W & 0 \\ 0 & 0 & 0 \end{pmatrix}. && (8.16\text{ -}68)
\end{aligned}$$

Finally, we have

$$\frac{\partial \mathbf{R}}{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}(t) = \mathbf{0}. \quad (8.16\text{ -}69)$$

We assemble the following matrix

$$\begin{aligned}
\mathbf{\Gamma}(t) &= \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial(RA, DEC, W)}(t) & \frac{\partial \boldsymbol{\omega}}{\partial(\dot{RA}, \dot{DEC}, \dot{W})}(t) \\ \frac{\partial \mathbf{R}}{\partial(RA, DEC, W)}(t) & \frac{\partial \mathbf{R}}{\partial(\dot{RA}, \dot{DEC}, \dot{W})}(t) \end{pmatrix}, \\
&= \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial(\phi, \theta, W)}(t) & \frac{\partial \boldsymbol{\omega}}{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}(t) \\ \frac{\partial \mathbf{R}}{\partial(\phi, \theta, W)}(t) & \frac{\partial \mathbf{R}}{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}(t) \end{pmatrix} \begin{pmatrix} \frac{\partial(\phi, \theta, W)}{\partial(RA, DEC, W)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}{\partial(\dot{RA}, \dot{DEC}, \dot{W})} \end{pmatrix}, \quad (8.16\text{ -}70)
\end{aligned}$$

where

$$\frac{\partial(\phi, \theta, W)}{\partial(RA, DEC, W)} = \frac{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}{\partial(\dot{RA}, \dot{DEC}, \dot{W})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8.16\text{ -}71)$$

and will use $\mathbf{\Gamma}(0)$ to chain the partial derivatives to $(RA(0), DEC(0), W(0), \dot{RA}(0), \dot{DEC}(0), \dot{W}(0))$ in order to estimate them.

We now focus on the partial derivatives of $(RA, DEC, W, \dot{RA}, \dot{DEC}, \dot{W})$ with respect to $\boldsymbol{\omega}$ and $\mathbf{R}$, although it is assumed that observations will only be functions of (RA, DEC, W). From eqs. 8.16 -55 and 8.16 -56 we see the full functional dependence on $\boldsymbol{\omega}$ and $\mathbf{R}$ to be

$$\boldsymbol{\alpha}(t, \mathbf{R}(t)) \equiv \begin{pmatrix} \phi(t, \mathbf{R}(t)) \\ \theta(t, \mathbf{R}(t)) \\ W(t, \mathbf{R}(t)) \end{pmatrix}, \tag{8.16 -72}$$

$$\begin{aligned} \dot{\boldsymbol{\alpha}}(t, \boldsymbol{\omega}(t), \mathbf{R}(t)) &\equiv \begin{pmatrix} \dot{\phi}(t, \boldsymbol{\omega}(t), \mathbf{R}(t)) \\ \dot{\theta}(t, \boldsymbol{\omega}(t), \mathbf{R}(t)) \\ \dot{W}(t, \boldsymbol{\omega}(t), \mathbf{R}(t)) \end{pmatrix}, \\ &= \mathbf{A}^{-1}(t, \mathbf{R}(t))\boldsymbol{\omega}(t). \end{aligned} \tag{8.16 -73}$$

It follows that

$$\begin{aligned} \frac{\partial(RA, DEC, W)}{\partial \boldsymbol{\omega}}(t) &= \frac{\partial(RA, DEC, W)}{\partial(\phi, \theta, W)} \frac{\partial(\phi, \theta, W)}{\partial \boldsymbol{\omega}}(t), \\ &= \mathbf{0}, \end{aligned} \tag{8.16 -74}$$

$$\begin{aligned} \frac{\partial(RA, DEC, W)}{\partial \mathbf{R}}(t) &= \frac{\partial(RA, DEC, W)}{\partial(\phi, \theta, W)} \frac{\partial(\phi, \theta, W)}{\partial \mathbf{R}}(t), \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{R_{23}}{R_{13}^2+R_{23}^2} & \frac{R_{13}}{R_{13}^2+R_{23}^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{1-R_{33}^2}} \\ 0 & 0 & \frac{R_{32}}{R_{31}^2+R_{32}^2} & 0 & 0 & -\frac{R_{31}}{R_{31}^2+R_{32}^2} & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{8.16 -75}$$

$$\begin{aligned} \frac{\partial(\dot{RA}, \dot{DEC}, \dot{W})}{\partial \boldsymbol{\omega}}(t) &= \frac{\partial(\dot{RA}, \dot{DEC}, \dot{W})}{\partial(\dot{\phi}, \dot{\theta}, \dot{W})} \frac{\partial(\dot{\phi}, \dot{\theta}, \dot{W})}{\partial \boldsymbol{\omega}}(t), \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A}^{-1}(t), \end{aligned} \tag{8.16 -76}$$

where R_{ij} is the element of $\mathbf{R}$ in the ith row and jth column. In order to understand how to compute $\frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \mathbf{R}}(t)$, we first compute one column corresponding to R_{ij} such that

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\alpha}}}{\partial R_{ij}}(t) &= -\mathbf{A}^{-1}(t)\frac{\partial \mathbf{A}}{\partial R_{ij}}(t)\mathbf{A}^{-1}(t)\boldsymbol{\omega}(t), \\
&= -\mathbf{A}^{-1}(t)\frac{\partial \mathbf{A}}{\partial R_{ij}}(t)\dot{\boldsymbol{\alpha}}(t), \\
&= -\mathbf{A}^{-1}(t)\left[\begin{array}{ccc} \frac{\partial \mathbf{A}}{\partial \phi}(t)\dot{\boldsymbol{\alpha}}(t) & \frac{\partial \mathbf{A}}{\partial \theta}(t)\dot{\boldsymbol{\alpha}}(t) & \frac{\partial \mathbf{A}}{\partial W}(t)\dot{\boldsymbol{\alpha}}(t) \end{array}\right]\left(\begin{array}{c} \frac{\partial \phi}{\partial R_{ij}}(t) \\ \frac{\partial \theta}{\partial R_{ij}}(t) \\ \frac{\partial W}{\partial R_{ij}}(t) \end{array}\right), \\
&= -\mathbf{A}^{-1}(t)\frac{\partial \boldsymbol{\omega}}{\partial(\phi,\theta,W)}(t)\left(\begin{array}{c} \frac{\partial \phi}{\partial R_{ij}}(t) \\ \frac{\partial \theta}{\partial R_{ij}}(t) \\ \frac{\partial W}{\partial R_{ij}}(t) \end{array}\right), \qquad (8.16\text{ -77})
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{\partial(\dot{RA},\dot{DEC},\dot{W})}{\partial \mathbf{R}}(t) &= \frac{\partial(\dot{RA},\dot{DEC},\dot{W})}{\partial(\dot{\phi},\dot{\theta},\dot{W})}\frac{\partial(\dot{\phi},\dot{\theta},\dot{W})}{\partial \mathbf{R}}(t), \\
&= -\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right)\mathbf{A}^{-1}(t)\frac{\partial \boldsymbol{\omega}}{\partial(\phi,\theta,W)}(t)\frac{\partial(\phi,\theta,W)}{\partial \mathbf{R}}(t) \qquad (8.16\text{ -78})
\end{aligned}$$

As a consistency check, we compute the partial derivatives of $\boldsymbol{\alpha}(t)$ and $\dot{\boldsymbol{\alpha}}(t)$ with respect to themselves at time t such that

$$\begin{pmatrix} \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\alpha}} & \frac{\partial \boldsymbol{\alpha}}{\partial \dot{\boldsymbol{\alpha}}} \\ \frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \boldsymbol{\alpha}} & \frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \dot{\boldsymbol{\alpha}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\omega}} & \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{R}} \\ \frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \boldsymbol{\omega}} & \frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \mathbf{R}} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\alpha}} & \frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\alpha}}} \\ \frac{\partial \mathbf{R}}{\partial \boldsymbol{\alpha}} & \frac{\partial \mathbf{R}}{\partial \dot{\boldsymbol{\alpha}}} \end{pmatrix},$$

$$= \begin{pmatrix} \mathbf{0} & \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{R}} \\ \left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\alpha}}}\right)^{-1} & -\left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\alpha}}}\right)^{-1} \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{R}} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\alpha}} & \frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\alpha}}} \\ \frac{\partial \mathbf{R}}{\partial \boldsymbol{\alpha}} & \mathbf{0} \end{pmatrix},$$

$$= \begin{pmatrix} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \boldsymbol{\alpha}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{8.16 -79}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{8.16 -80}$$

where we use the properties that $\frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \boldsymbol{\omega}} = \left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\alpha}}}\right)^{-1}$, and $\frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \boldsymbol{\alpha}} = \mathbf{I}$. It follows that the same relationship holds between the partial derivatives of $(RA, DEC, W, \dot{RA}, \dot{DEC}, \dot{W})$ with respect to themselves.

Finally, we form the matrix

$$\boldsymbol{\Lambda}(t) = \left[\frac{\partial(RA, DEC, W)}{\partial \boldsymbol{\omega}}(t) \quad \frac{\partial(RA, DEC, W)}{\partial \mathbf{R}}(t) \right],$$

$$= \left[\mathbf{0} \quad \frac{\partial(RA, DEC, W)}{\partial \mathbf{R}}(t) \right], \tag{8.16 -81}$$

and use it to construct the partial derivatives of interest

$$\frac{\partial(RA, DEC, W)}{\partial(RA_0, DEC_0, W_0, \dot{RA}_0, \dot{DEC}_0, \dot{W}_0)}(t) = \boldsymbol{\Lambda}(t)\boldsymbol{\Omega}(t)\boldsymbol{\Gamma}(0). \tag{8.16 -82}$$

These can be chained with any observable that is a function of (RA, DEC, W) in order to estimate the initial conditions $(RA(0), DEC(0), W(0), \dot{RA}(0), \dot{DEC}(0), \dot{W}(0))$. One final note is that we prefer to integrate $\boldsymbol{\Omega}(t)$ through time rather than $\frac{\partial(RA, DEC, W)}{\partial(RA_0, DEC_0, W_0, \dot{RA}_0, \dot{DEC}_0, \dot{W}_0)}(t)$ because of the possible discontinuities inherent in the angular values when using the latter.

8.16.5 Application to computing partials of angular velocity and reference frame with respect to moments of inertia through time

In addition to estimating the initial conditions of the angular velocity and reference frame we may also be interested in determining the moments of inertia, that is, the independent elements of the moment of inertia tensor in the body-fixed frame. This then requires the computation of partial derivatives of angular velocity and reference frame with respect to these elements through time. Therefore, we use eq. 8.16 -4 and need to determine the matrix $\mathbf{G}(t)$. Because the moment of inertia tensor is a symmetric, positive-definite matrix, we only consider the six independent elements, which we denote as the vector $\mathbf{s}$, such that in the body-fixed frame

$$\mathbf{I}_c = \begin{pmatrix} s(1) & s(4) & s(6) \\ s(4) & s(2) & s(5) \\ s(6) & s(5) & s(3) \end{pmatrix}. \tag{8.16 -83}$$

Thus, the matrix $\mathbf{G}(t)$ is given by

$$\mathbf{G}(t) = \begin{pmatrix} \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{s}}(t) \\ \\ \frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{s}}(t) \end{pmatrix}. \tag{8.16 -84}$$

8.16.5.1 Computing $\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{s}}(t)$ We begin by taking the partial derivative of eq. 8.16 -8 with respect to a single element s_j. We use the fact that for any invertible matrix $\mathbf{A}(\alpha)$ that is a function of a scalar α, we have

$$\frac{\partial \mathbf{A}^{-1}}{\partial \alpha} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{A}^{-1}. \tag{8.16 -85}$$

This gives

$$\begin{aligned} \frac{\partial \dot{\boldsymbol{\omega}}}{\partial s_j}(t) &= -\mathbf{I}_c^{-1}\frac{\partial \mathbf{I}_c}{\partial s_j}\mathbf{I}_c^{-1}\left[\mathbf{T}(t) - \boldsymbol{\omega}(t) \times \mathbf{I}_c\boldsymbol{\omega}(t)\right] - \mathbf{I}_c^{-1}\left[\boldsymbol{\omega}(t) \times \frac{\partial \mathbf{I}_c}{\partial s_j}\boldsymbol{\omega}(t)\right], \\ &= -\mathbf{I}_c^{-1}\left[\frac{\partial \mathbf{I}_c}{\partial s_j}\dot{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}(t) \times \frac{\partial \mathbf{I}_c}{\partial s_j}\boldsymbol{\omega}(t)\right], \\ &= -\mathbf{I}_c^{-1}\left[\mathbf{I}\frac{\partial \mathbf{I}_c}{\partial s_j}\dot{\boldsymbol{\omega}}(t) + \mathbf{E}_{\boldsymbol{\omega}}(t)\frac{\partial \mathbf{I}_c}{\partial s_j}\boldsymbol{\omega}(t)\right], \\ &= -\mathbf{I}_c^{-1}\left[\left(\dot{\boldsymbol{\omega}}^{\mathrm{T}}(t) \otimes \mathbf{I}\right) + \left(\boldsymbol{\omega}^{\mathrm{T}}(t) \otimes \mathbf{E}_{\boldsymbol{\omega}}(t)\right)\right] vec\left(\frac{\partial \mathbf{I}_c}{\partial s_j}\right). \end{aligned} \tag{8.16 -86}$$

This leads to

$$\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{s}}(t) = -\mathbf{I}_c^{-1}\left[\left(\dot{\boldsymbol{\omega}}^{\mathrm{T}}(t) \otimes \mathbf{I}\right) + \left(\boldsymbol{\omega}^{\mathrm{T}}(t) \otimes \mathbf{E}_{\boldsymbol{\omega}}(t)\right)\right]\frac{\partial \mathbf{I}_c}{\partial \mathbf{s}}, \tag{8.16 -87}$$

where

$$\frac{\partial \mathbf{I}_c}{\partial \mathbf{s}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{8.16 -88}$$

8.16.5.2 Computing $\frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{s}}(t)$ We see from eq. 8.16 -6 that $\dot{\mathbf{R}}(t)$ is not a function of $\mathbf{I}_c$ and so

$$\frac{\partial \dot{\mathbf{R}}}{\partial \mathbf{s}}(t) = \mathbf{0}. \tag{8.16 -89}$$

8.16.5.3 Variational equations We now wish to compute the partial derivatives of angular velocity and the reference frame with respect to moments of inertia analogous to eq. 8.16 -4, and so

$$\begin{cases} \mathbf{\Psi}(t) &= \int_0^t \left[\mathbf{F}(\tau)\mathbf{\Psi}(\tau) + \mathbf{G}(\tau)\right] d\tau \\ \dot{\mathbf{\Psi}}(t) &= \mathbf{F}(t)\mathbf{\Psi}(t) + \mathbf{G}(t), \quad \mathbf{\Psi}(0) = \mathbf{0} \end{cases}, \tag{8.16 -90}$$

where

$$\mathbf{\Psi}(t) = \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{s}}(t) \\ \frac{\partial \mathbf{R}}{\partial \mathbf{s}}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{s}}(t) \\ \mathbf{0} \end{pmatrix}, \tag{8.16 -91}$$

and $\mathbf{F}(t)$ is the same as in eq. 8.16 -51.

8.16.5.4 Torque-free motion case In the case of torque-free motion, eq. 8.16 -7 becomes

$$\mathbf{0} = \mathbf{I}_c \dot{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}(t) \times \mathbf{I}_c \boldsymbol{\omega}(t). \tag{8.16 -92}$$

This equation is enforced at each time step during the integration process. Now let $\mathbf{s}$ contain the six independent elements of $\mathbf{I}_c$ in eq. 8.16 -92. If we now multiply eq. 8.16 -87 by this $\mathbf{s}$ vector, we get

$$\begin{aligned}
\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \mathbf{s}}(t)\mathbf{s} &= -\mathbf{I}_c^{-1}\left[\left(\dot{\boldsymbol{\omega}}^{\mathrm{T}}(t)\otimes\mathbf{I}\right)+\left(\boldsymbol{\omega}^{\mathrm{T}}(t)\otimes\mathbf{E}_{\boldsymbol{\omega}}(t)\right)\right]\frac{\partial \mathbf{I}_c}{\partial \mathbf{s}}\mathbf{s},\\
&= -\mathbf{I}_c^{-1}\left[\left(\dot{\boldsymbol{\omega}}^{\mathrm{T}}(t)\otimes\mathbf{I}\right)+\left(\boldsymbol{\omega}^{\mathrm{T}}(t)\otimes\mathbf{E}_{\boldsymbol{\omega}}(t)\right)\right] vec\left(\mathbf{I}_c\right),\\
&= -\mathbf{I}_c^{-1}\left[\mathbf{I}_c\dot{\boldsymbol{\omega}}(t)+\mathbf{E}_{\boldsymbol{\omega}}(t)\mathbf{I}_c\boldsymbol{\omega}(t)\right],\\
&= -\mathbf{I}_c^{-1}\left[\mathbf{I}_c\dot{\boldsymbol{\omega}}(t)+\boldsymbol{\omega}(t)\times\mathbf{I}_c\boldsymbol{\omega}(t)\right],\\
&= -\mathbf{I}_c^{-1}\mathbf{0},\\
&= \mathbf{0},
\end{aligned} \tag{8.16 -93}$$

where eq. 8.16 -92 has been used. This means that $\mathbf{G}(t)\mathbf{s}=\mathbf{0}$ for all times t, as long as $\mathbf{s}$ reflects the current values of $\mathbf{I}_c$ that are used in eq. 8.16 -92.

Now, let us consider $\boldsymbol{\Psi}(t)$ in eq. 8.16 -90. Although in practice the indicated integration is carried out with sophisticated methods, it will suffice to use the simple Euler forward method as the canonical form. For our case, it is given by

$$\begin{aligned}
\boldsymbol{\Psi}(t_n) &= \boldsymbol{\Psi}(t_{n-1})+\Delta t_n\dot{\boldsymbol{\Psi}}(t_{n-1},\boldsymbol{\Psi}(t_{n-1})),\\
&= \boldsymbol{\Psi}(t_{n-1})+\Delta t_n\left[\mathbf{F}(t_{n-1})\boldsymbol{\Psi}(t_{n-1})+\mathbf{G}(t_{n-1})\right],\\
&= \left[\mathbf{F}(t_{n-1})\Delta t_n+\mathbf{I}\right]\boldsymbol{\Psi}(t_{n-1})+\mathbf{G}(t_{n-1})\Delta t_n,
\end{aligned} \tag{8.16 -94}$$

where $\Delta t_n = t_n - t_{n-1}$. We will now prove by induction that $\boldsymbol{\Psi}(t)\mathbf{s}=\mathbf{0}$ for all times t.

Theorem 8.1. *If*

$$\boldsymbol{\Psi}(t_n) = \begin{pmatrix} \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{s}}(t_n) \\ \\ \frac{\partial \mathbf{R}}{\partial \mathbf{s}}(t_n) \end{pmatrix},$$

and $\mathbf{s}$ *contains the independent elements of* $\mathbf{I}_c$*, then* $\boldsymbol{\Psi}(t_n)\mathbf{s}=\mathbf{0}$ *for all* n*.*

Proof. We are given that $\boldsymbol{\Psi}(t_0)=\mathbf{0}$ and $\mathbf{G}(t_n)\mathbf{s}=\mathbf{0}$ for all n.

1. Prove true for $n=1$: Here we have from the Euler forward method

$$\begin{aligned}
\boldsymbol{\Psi}(t_1) &= \left[\mathbf{F}(t_0)\Delta t_1+\mathbf{I}\right]\boldsymbol{\Psi}(t_0)+\mathbf{G}(t_0)\Delta t_1,\\
&= \mathbf{G}(t_0)\Delta t_1.
\end{aligned}$$

   Therefore, $\boldsymbol{\Psi}(t_1)\mathbf{s}=\mathbf{0}$.

2. Assume true for n: Therefore, $\boldsymbol{\Psi}(t_n)\mathbf{s}=\mathbf{0}$.

3. Prove true for $n+1$: Here we have from the Euler forward method

$$\boldsymbol{\Psi}(t_{n+1}) = \left[\mathbf{F}(t_n)\Delta t_{n+1}+\mathbf{I}\right]\boldsymbol{\Psi}(t_n)+\mathbf{G}(t_n)\Delta t_{n+1}.$$

Therefore, from the previous assumption

$$\begin{aligned}\mathbf{\Psi}(t_{n+1})\mathbf{s} &= [\mathbf{F}(t_n)\Delta t_{n+1} + \mathbf{I}]\,\mathbf{\Psi}(t_n)\mathbf{s} + \mathbf{G}(t_n)\mathbf{s}\Delta t_{n+1}. \\ &= \mathbf{0}\end{aligned}$$

□

This is further corroborated by the torque-free form of eq. 8.16 -8 which is seen to be invariant to a scale factor α on the moment of inertia tensor, that is

$$\begin{aligned}\dot{\boldsymbol{\omega}}(t) &= -\left(\alpha \mathbf{I}_c\right)^{-1}\left[\boldsymbol{\omega}(t) \times \left(\alpha \mathbf{I}_c\right)\boldsymbol{\omega}(t)\right], \\ &= -\mathbf{I}_c^{-1}\left[\boldsymbol{\omega}(t) \times \mathbf{I}_c\boldsymbol{\omega}(t)\right]. \end{aligned} \qquad (8.16\ \text{-}95)$$

This invariance corresponds precisely to the null-space of $\mathbf{\Psi}(t)$ shown in Theorem 8.1. Therefore, when dealing with torque-free or negligible torque cases, it appears that the initial orientation angles and their rates can be observed along with all ratios of elements of the moment of inertia tensor in any body-fixed frame. That is, we can fix a single diagonal element, say $\mathbf{I}_c(1,1) = 1$ and solve for all others relative to this value. The absolute levels of these elements can be adjusted via comparison with a gravity analysis.

9 INTEGRATION AND INTERPOLATION

GEODYN uses Cowell's Sum method for direct numerical integration of both the equations of motion and the variational equations to obtain the position and velocity and the attendant variational partials at each observation time. The integrator output is not required at actual observation times; it is output on an even integration step. GEODYN uses an interpolation technique to obtain values at the actual observation time. The specific numerical methods used in GEODYN for this integration and interpolation are presented below. These procedures are controlled by subroutine ORBIT.

9.1 INTEGRATION

Let us first consider the integration of the equations of motion. These equations are three second order differential equations in position! and may be formulated as six first order equations in position and velocity if a first order integration scheme were used for their solution. For reasons of increased accuracy and stability, the position vector $\bar{r}$ is obtained by a second order integration of the accelerations $\ddot{\bar{r}}$,whereas as the velocity vector $\dot{\bar{r}}$ is obtained as the solution of a first order system. These are both multi-step methods requiring at least one derivative evaluation on each step.

The integration scheme is equivalent to the interpolator with arguments 1 and 0 for predictor and corrector respectively.

To integrate the position components, the predictor

$$\bar{r}_{n+1} = (S_2 + \sum_{p=0}^{q} \gamma_p^* \ddot{\bar{r}}_{n-p})h^2 \tag{9.1 -1}$$

is applied, followed by a Cowell corrector:

$$\bar{r}_{n+1} = (S_2 + \sum_{p=0}^{q} \gamma_p \ddot{\bar{r}}_{n-p+1})h^2 \tag{9.1 -2}$$

The velocity components are integrated using the predictor:

$$\dot{\bar{r}}_{n+1} = (S_1 + \sum_{p=0}^{q+1} \beta_p^* \ddot{\bar{r}}_{n-p})h \tag{9.1 -3}$$

followed by an Adams-Moulton corrector;

$$\dot{\bar{r}}_{n+1} = (S_1 + \sum_{p=0}^{q+1} \beta_p \ddot{\bar{r}}_{n-p+1})h \tag{9.1 -4}$$

in these integration formulae, It is the integration step size, q has the value ORDER-2, and γ_p, γ_p^*, β_p, β_p^* are the precomputed coefficients.

Let us next consider the integration of the variational equations. Following section 8.2, these equations may be written as

$$\ddot{Y} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} Y \\ \dot{Y} \end{bmatrix} + f \tag{9.1 -5}$$

where (recall equation 8.2-7)

$$\begin{bmatrix} Y \\ \dot{Y} \end{bmatrix} = X_m$$

and, partitioning according to position and velocity partials,

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} U_{2c} + D_r \end{bmatrix}$$

where U_{2c} is defined in equation 8.3-20.

Because A, B and f are functions only of the orbital parameters, the integration can be and is performed using only corrector formulae, (Note that A, B and f must be evaluated with the final corrected values of $\bar{r}_{n+1}$ and $\dot{\bar{r}}_{n+1}$)

In the above corrector formulae, we substitute the equation for $\ddot{Y}$ and solve explicitly for Y and $\dot{Y}$:

$$\begin{bmatrix} Y_{n+1} \\ \dot{Y}_{n+1} \end{bmatrix} = (I - H)^{-1} \begin{bmatrix} \bar{X}_n \\ \bar{V}_n \end{bmatrix} \tag{9.1 -6}$$

Under certain conditions, a reduced form of this solution is used. It can be seen from the variational and observation equations that if drag is not a factor and there are no range rate, doppler, or altimeter rate measurements, the velocity variational partials are not used. There is then no need to integrate the velocity variational equations. This represents a significant time saving. In the integration algorithm, the B matrix is zero and $(l - J)$ is reduced to a three by three matrix.

A detailed description of the H matrix and the $\bar{X}_n$ and $\bar{V}_n$ and vectors can be found in pages 16,17 of [9 - 2].

Backwards integration involves only a few simple modifications to these normal or forward integration procedures. These modifications are to negate the step size, and invert the time completion test.

9.2 THE INTEGRATOR STARTING SCHEME

The predictor-corrector combination employed to proceed with the main integration is not self-starting. That is, each step of the integration requires the knowledge of past values of the solution that are not available at the start of the integration. Thus, a separate procedure for starting the integration is needed. The method presented here is that implemented In GEODYN.

A Taylor series approximation is used to predict initial values of position and velocity. With these ?starting values, the Sum array is evaluated using epoch positions and velocities. Now the loop is closed by interpolations for the positions and velocities not at epoch and their accelerations evaluated. The Sums are now again evaluated, this procedure continues until the Sums COD verge to the desired accuracy.

9.3 INTERPOLATION

GEODYN uses interpolation for two functions. The first is the interpolation of the orbit elements and variational partials to the observation times; the second is the interpolation for mid-points when the integrator is decreasing the step size (used only with a variable step integrator - not currently avaliable). The formulas used by INTRP are:

$$X(t+\Delta t) = \left[S_2(t) + \left[\frac{\Delta t}{h} - 1\right] S_1(t) + \sum_{t=0}^{n} C_i(\Delta t) f_{n-i}\right] h^2 \qquad (9.3\ \text{-1})$$

for positions and

$$\dot{X}(t+\Delta t) = \left[S_1(t) + \sum_{t=0}^{n} C_i'(\Delta t) f_{n-i}\right] h \qquad (9.3\ \text{-2})$$

for velocities S_1 and S_2 are the first and second sums carried along by the integrator, f's are the back values of acceleration, h the step size, and C_i, C_i' are the interpolation coefficients.

10 THE STATISTICAL ESTIMATION SCHEME

The basic problem in orbit determination is to calculate, from a given set of observations of the spacecraft, a set of parameters specifying the trajectory of a spacecraft. Because there are generally more observations than parameters, the parameters are overdetermined. Therefore, a statistical estimation scheme is necessary to estimate the "best" set of parameters.

The estimation scheme selected for GEODYN is a partitioned Bayesian least squares method. The complete development of this procedure is presented in this section.

It should be noted that the functional relationships between the observations and parameters are in general non-lineal'; thus an iterative procedure is necessary to solve the resultant non-linear normal equations. The Newton-Raphson iteration formula is used to solve these equations.

10.1 BAYESIAN LEAST SQUARES ESTIMATION

Consider a vector of N independent observations $\vec{z}$ whose values can be expressed as known functions of M parameters denoted by the vector $\vec{x}$. The following non-linear regression equation holds:

$$\vec{z} = \vec{f}(\vec{x}) + \vec{\sigma} \tag{10.1 -1}$$

where $\vec{\sigma}$ is the N vector denoting the noise on the observations. Given $\vec{z}$,the functional form of $\vec{f}$, and the statistical properties of $\vec{\sigma}$ we must obtain the estimate of $\vec{x}$ that is "best" in some sense.

Bayes theorem in probability holds for probability density functions and can be written as follows:

$$p(\vec{x}|\vec{z}) = \frac{p(\vec{x})}{p(\vec{z})} p(\vec{z}|\vec{x}) \tag{10.1 -2}$$

where

$p(\vec{x}|\vec{z})$ is the joint conditional probability density function for the parameter vector $\vec{x}$, given that the data vector $\vec{z}$ has occurred;

$p(\vec{x})$ is the joint probability density function for the vector $\vec{x}$;

$p(\vec{z})$ is the joint probability density function for the vector $\vec{z}$;

$p(\vec{z}|\vec{x})$ is the joint conditional density function for the parameter vector $\vec{x}$, given that the data vector $\vec{z}$ has occurred;

$p(\vec{x})$ is often referred to as the a priori density function of $\vec{x}$, and $p(\vec{x}|\vec{z})$ is referred to as the a posteriori conditional density function. In any Bayesian estimation scheme, we must determine this a posteriori density function and from this function determine a "best" estimate of $\vec{x}$ which can be denoted $\hat{\vec{x}}$

To obtain the a posteriori conditional density function, we must make an assumption concerning the statistical properties of the on the observations: the noise vector $\vec{\sigma}$ has a joint normal distribution with mean vector $\vec{O}$ and a variance-covariance matrix Σ_z. Σ_2 is an $N \times N$ matrix and is assumed diagonal, that is, the observations are considered to be independent and uncorrelated. The "best" estimate of $\vec{x}$, $\hat{\vec{x}}$ is defined as that vector maximizing the a posteriori density function; this is equivalent to choosing the mean value of this distribution. An estimator of this type has been referred to as the maximum likelihood estimate in the Bayesian sense [10.2].

A further assumption is that the a priori density function $p(\hat{\vec{x}})$ is a joint normal distribution and is written as follows:

$$p(\hat{\vec{x}}) = \left[\frac{det[\Sigma_A^{-1}]}{2\pi^M}\right]^{\frac{1}{2}} exp\left[-\frac{1}{2}(\vec{x}_A - \hat{\vec{x}})^T \Sigma_A^{-1} (\vec{x}_A - \hat{\vec{x}})\right] \tag{10.1 -3}$$

where

$\vec{x}_A$ is the a priori estimate of the parameter vector,

Σ_A is the a priori variance-covariance matrix associated with the a priori parameter vector, $\vec{x}_A$

Σ_A is an M x M matrix, which may or may not be diagonal.

The conditional density function $p(\vec{z}|\vec{x})$ can be written as follows

$$p(\vec{z}|\hat{\vec{x}}) = \left[\frac{det[\Sigma_z^{-1}]}{2\pi^N}\right]^{\frac{1}{2}} exp - \frac{1}{2}\left[(\vec{z} - f(\hat{\vec{x}})\right]^T \Sigma_z^{-1}\left[(\vec{z} - f(\hat{\vec{x}})\right] \qquad (10.1\text{ -4})$$

where Σ_z is the variance-covariance matrix associated with the data vectors, $\vec{z}$.

It can be shown from equation 10.1-2 that maximizing the a posteriori density function $p(\hat{\vec{x}}|\vec{z})$ is equivalent to maximizing the product $p(\hat{\vec{x}})p(\vec{z}|\hat{\vec{x}})$ because the density function $p(\vec{z})$ is a constant valued function. Further, this reduces to minimizing the following quadratic form:

$$\left[\vec{x}_A - \hat{\vec{x}}\right]^T \Sigma_a^{-1}\left[\vec{x}_A - \hat{\vec{x}}\right] + \left[\vec{z} - f(\hat{\vec{x}})\right]^T \Sigma_z^{-1}\left[\vec{x} - f(\hat{\vec{x}})\right] \qquad (10.1\text{ -5})$$

This results in the following set of M non-linear equations:

$$B^T\Sigma_z^{-1}\left[\vec{z} - f(\hat{\vec{x}})\right]^T + \Sigma_A^{-1}\left[\vec{x}_A - \hat{\vec{x}}\right] \qquad (10.1\text{ -6})$$

where B is an $N \times M$ matrix with elements

$$B_{NM} = \frac{\partial f_N(\hat{\vec{x}})}{\partial x_M}|\vec{x} = \hat{\vec{x}}$$

This equation defines the Bayesian least squares estimation procedure. We have not stated how the a priori parameter vector and variance-covariance matrix were obtained. In practice these a priori values are almost always estimates that have been obtained from some previous data. In these cases the Bayesian estimates are identical to the classical maximum likelihood estimates that would be obtained if all the data were used; in this context the a priori parameters can be considered as additional observations.

The variance-covariance matrix of $\hat{\vec{x}}, V$, is given by the following formula:

$$V = [B^T\Sigma_z^{-1} + \Sigma_A^{-1}]^{-1} \qquad (10.1\text{ -7})$$

Solution of the Estimation Formula

Equation (10.1-6) defines a set of M non-linear equations in M unknowns $\hat{\vec{x}}$; these equation are solved using the Newton-Raphson iteration formula. Equation (10.2-6) can be written as follows:

$$\vec{F}(\hat{\vec{x}}) = 0$$

The iteration formula is

$$\hat{\vec{x}}^{n+1} = \hat{\vec{x}} - \left[\frac{\partial \vec{F}(\hat{\vec{x}})}{\partial \hat{\vec{x}}}\right]^{-1} \vec{F}(\hat{\vec{x}}^{(n)}) \tag{10.1 -8}$$

where

$\hat{\vec{x}}^{(n)}$ is the nth approximation to the true solution $\hat{\vec{x}}$.

Now

$$F(\hat{\vec{x}}) = B^T \Sigma_z^{-1} \left[\vec{z} - f(\hat{\vec{x}})\right] + \Sigma_A^{-1} \left[\vec{x}_A - \hat{\vec{x}}\right] = 0 \tag{10.1 -9}$$

Then differentiating and neglecting second derivatives,

$$\left[\frac{\partial \vec{F}(\hat{\vec{x}})}{\partial \hat{\vec{x}}}\right] = B^T \Sigma_z^{-1} \tag{10.1 -10}$$

Substituting equation (10.1-10) in equation (10.1-8) gives

$$\hat{\vec{x}}^{n+1} - \hat{\vec{x}}^{(n)} = [B^T \Sigma_z^{-1} B + \Sigma_A^{-1}]^{-1} \left[B^T \Sigma_z^{-1} [\vec{z} - f(\hat{\vec{x}}^{(n)})] + \Sigma_A^{-1} [\vec{x}_A - \hat{\vec{x}}^{(n)}]\right] \tag{10.1 -11}$$

Now let $\hat{\vec{x}}^{(n+1)} - \hat{\vec{x}}^{(n)}$, the correction to the n^{th} approximation, be denoted by $d\vec{x}^{(n+1)}$, and let $\vec{z} - f(\hat{\vec{x}}^{(n)})$, the vector of residuals from the n^{th} approximation, be $d\vec{x}^{(n)}$. Equation (10.1-11) becomes

$$d\vec{x}^{(n+1)} = [B^T \Sigma_z^{-1} B + \Sigma_A^{-1}]^{-1} \left[B^T \Sigma_z^{-1} d\vec{x}^{(n)} + \Sigma_A^{-1} [\vec{x}_A - \hat{\vec{x}}^{(n)}]\right] \tag{10.1 -12}$$

10.2 THE PARTITIONED SOLUTION

In a multi-satellite, multi-arc estimation program such as GEODYN, it is necessary for reasons of memory size to formulate the estimation scheme in a manner such that the information for all satellite arcs are not in memory simultaneously. The procedure used in GEODYN is a partitioned Bayesian Estimation Scheme which requires only common parameter information and the information for a single arc to be in memory at any given time. The development of the GEODYN partitioned solution is given here.

The Bayesian estimation formula has been developed in the previous section as

$$d\vec{x}^{(n+1)} = [B^T W B + V_A^{-1}]^{-1} \left[B^T W d\vec{m} + V_A^{-1} [\vec{x}_A - \hat{\vec{x}}^{(n)}] \right] \quad (10.2\text{ -1})$$

where

- $\vec{x}_A$ is the a priori estimate of $\vec{x}$.
- V_A is the a priori covariance matrix. associated with $\vec{x}_A$. (called Σ_A in section 10.1)
- W is the weighting matrix associated with the observations. (called Σ_z^{-1} in section 10.1)
- $\vec{x}^{(n)}$ is the nth approximation to the true solution $\hat{\vec{x}}$,
- $d\vec{m}$ is the vector of residuals (0-C) from the n^{th} approximation. (called $dz^{(n)}$ in section 10.1)
- $d\vec{x}^{(n+1)}$ is the vector of corrections to the parameters; i.e., $\vec{x}^{(n+1)} = \vec{x}^{(n)} + d\vec{x}^{(n+1)}$
- B is the matrix of partial derivatives of the observations with respect to the parameters were the i, j^{th}, is given by $\frac{\partial m_i}{\partial x_i}$

The iteration formula given by this equation solves the non-linear normal equations formed by minimizing the sum of squares of the weighted residuals.

We desire a solution wherein $\vec{x}$ is partitioned according to $\vec{a}$, the vector of parameters associated only with individual arcs; and $\vec{k}$, the vector of parameters common to all arcs. For geodetic parameter estimation $\vec{k}$ consists of the geopotential coefficients and station coordinates.

As a result of this partitioning, we may write B, the matrix of partial derivatives of the observations, as

$$B = [B_a, B_k] \quad (10.2\text{ -2})$$

where

$$[B_a]_{i,j} = \frac{\partial m_i}{\partial a_j}$$
$$[B_k]_{i,j} = \frac{\partial m_i}{\partial k_j}$$

We may also write V_A, the covariance matrix of the parameters as

$$V_A = \begin{bmatrix} V_a & 0 \\ 0 & V_k \end{bmatrix} \tag{10.2 -3}$$

where we have assumed the independence of the a priori information on the arc parameters and common parameters (in practice valid to an extremely high degree).

We may now rewrite our iteration formula:

$$\begin{bmatrix} d\vec{a} \\ d\vec{k} \end{bmatrix} = \begin{bmatrix} B_a^T W B_a + V_a^{-1} & B_a^T W B_k \\ [B_a^T W B_k]^T & B_k^T W B_k + V_k^{-1} \end{bmatrix}^{-1} \times \begin{bmatrix} B_a^T W dm^{(n)} + V_a^{-1}(\vec{a}^{(n)} - \vec{a}_A) \\ B_k^T W dm^{(n)} + V_k^{-1}(\vec{k}^{(n)} - \vec{k}_A) \end{bmatrix}$$
$$= \begin{bmatrix} A & A_k \\ A_k^T & K \end{bmatrix}^{-1} \begin{bmatrix} C_a \\ C_k \end{bmatrix} \tag{10.2 -4}$$

The required matrix inversion is obtained by partitioning. We write

$$\begin{bmatrix} N_1 & N_2 \\ N_2^T & N_4 \end{bmatrix} \begin{bmatrix} A & A_k \\ A_k^T & K \end{bmatrix} = I \tag{10.2 -5}$$

and, solving the resulting equations, determine

$$N_1 = A^{-1} = [A^{-1} A_k] N_4 [A_k^T A^{-1}] \tag{10.2 -6}$$
$$N_2 = -A^{-1} A_k N_4 \tag{10.2 -7}$$

and

$$N_4 = [K - A_k^T A^{-1} A_k]^{-1} \tag{10.2 -8}$$

There is no problem associated with inverting A because the existence of the a information alone guarantees that the inverse of A exists. On the other hand, the inverse of $K - A_k^T A^{-1} A_k$ is not guaranteed to exist. High correlations between the parameters could make the matrix nearly singular. In practice, however, the use of a reasonable amount of a priori information eliminates any inversion difficulties.

The iteration formula may now be written as

$$\begin{bmatrix} d\vec{a} \\ d\vec{k} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^T & N_4 \end{bmatrix} \begin{bmatrix} C_a \\ C_k \end{bmatrix} \tag{10.2 -9}$$

or

$$d\vec{a} = [A^{-1} + (A^{-1}A_k)N_4(A_k^T A^{-1})]C_a - A^{-1}A_k N_4 C_k \tag{10.2 -10}$$

$$d\vec{k} = -N_4 A_k^T A^{-1} C_a + N_4 C_k \tag{10.2 -11}$$

Noting the similarities between $d\vec{a}$ and $d\vec{k}$ we write

$$d\vec{a} = A^{-1}C_a - A^{-1}A_k d\vec{k} \tag{10.2 -12}$$

$$d\vec{k} = N_4(C_k - A_k^t A^{-1} C_a) \tag{10.2 -13}$$

Note that most of the elements of A are zero because the measurements in any given arc are independent of the arc parameters of any other arc. Also, the covariances between the a priori information associated with each arc are assumed to be zero. Thus, both A and V_a are composed of zeroes except for matrices, A_r and V_r respectively, along the diagonal, where r is a subscript denoting the r_{th} arc,

e. g., a_r

$$[A_r]_{i,j} = \sum_l \frac{\partial m_i}{\partial a_{r_i}} \frac{1}{\sigma_l^2} \frac{\partial m_i}{\partial a_{r_j}} + [V_r^{-1}]_{i,j} \tag{10.2 -14}$$

where

- l ranges over the measurements in the r^{th} arc and i, j range over the parameters in the r^{th} arc, $\vec{a}_r$
- V_r is the partition of V_a associated with the r^{th} arc.

The reader should note that A^{-1}, like A, is composed of zeroes except for matrices A_r^{-1} along the diagonal.

We shall also require the partitions of A_k and C_a according to each arc. These partitions are given by

$$[A_{rk}]_{i,j} = \sum_l \frac{\partial m_l}{\partial a_{r_i}} \frac{1}{\sigma_l^2} \frac{\partial m_l}{\partial k_j} \tag{10.2 -15}$$

and

$$[C_r]_i = \sum_l \frac{\partial m_l}{\partial a_{r_i}} \frac{1}{\sigma_l^2} d\vec{m}_l \tag{10.2 -16}$$

where the subscript r again denotes the r^{th} arc and I ranges over the measurement partials and residuals in the r^{th} arc.

Let us now investigate the matrix partitions in the solutions for $d\vec{a}$ and $d\vec{k}$. We consider A^{-1} to be a diagonal matrix with diagonal elements A_r^{-1} and C_a to be a column vector with elements C_r . Hence

$$[A^{-1}C_a] = A_r^{-1}C_r \tag{10.2 -17}$$

is the r^{th} element of the product matrix. A_k is considered to be a column vector with elements A_{rk}, thus

$$[A_k^T A^{-1} C_a] = A_{rk}^T A_r^{-1} C_r \tag{10.2 -18}$$

The elements in the product $A^{-1}A_k$ are given by

$$[A^{-1}A_k]_r = A_r^{-1}A_{rk} \tag{10.2 -19}$$

We also require the product $A_k^T A^{-1} A_k$. Its elements are given by

$$[A_k^T A^{-1} A_k]_{r,r} = A_{rk}^T A_r^{-1} A_{rk} \tag{10.2 -20}$$

Thus,

$$d\vec{a}_r = A_r^{-1}C_r - A_r^{-1}A_{rk}d\vec{k} \tag{10.2 -21}$$

$$d\vec{k} = N_4[C_k - \sum_r A_{rk}^T A_r^{-1} C_r] \tag{10.2 -22}$$

where

$$N_4 = [K - \sum_r A_{rk}^T A_r^{-1} A_{rk}]^{-1} \tag{10.2 -23}$$

These solutions form the partitioned Bayesian estimation scheme used in GEODYN.

Additionally, after all arcs have been independently adjusted, and then the common parameters have been adjusted, the covariance matrix. for the arc parameters must be updated to account for the simultaneous adjustment of the common parameters:

$$[N_1]_r = A_r^{-1} + [A_r^{-1}A_{rk}]N_4[A_{rk}^T A_r^{-1}] \tag{10.2 -24}$$

Summary

The procedure for computer implementation is illustrated in Figure 1. This procedure is:

1. Integrate through each arc fanning the matrices A_r, A_{rk}, and C_r; and simultaneously accumulate into the common parameter matrices K and C_k.

2. At the end of each are, form

$$d\vec{a'_r} = A_r^{-1} C_r \qquad (10.2\text{ -25})$$

   and modify the common parameter matrices as follows:

$$K = K - A_{rk}^T A_r^{-1} A_{rk} \qquad (10.2\text{ -26})$$

   and

$$C_k = C_k - A_{rk} d\vec{a'}_r \qquad (10.2\text{ -27})$$

   The matrices $d\vec{a'}_r, A_{rk}$, and A_r^{-1} must also be put in external storage.

3. After processing all of the arcs; i.e., at the end of a global or "outer" iteration, determine $d\vec{k}$. Note that K has become N_4^{-1} and C_k has been modified so that

$$d\vec{k} = K^{-1} C_k \qquad (10.2\text{ -28})$$

   The updated values for the common parameters are of course given by

$$\vec{k}^{(n+l)} = \vec{k}^{(n)} + d\vec{k} \qquad (10.2\text{ -29})$$

   The arc parameters are then updated to account for the simultaneous solution of the common parameters. Information for each arc is input in turn; that is, the previously stored $d\vec{a'}_r, A_{rk}$, and A_r^{-1} are read into memory, The correction vector to the updated arc parameters is given by

$$d\vec{a}_r = d\vec{a'}_r - (A_r^{-1} A_{rk}) d\vec{k} \qquad (10.2\text{ -30})$$

   and hence

$$\vec{a}_r^{(n+1)} = \vec{a}_r^{(n)} + d\vec{a}_r \qquad (10.2\text{ -31})$$

   The covariance matrix for the arc parameters, A_r^{-1}, is updated by

$$A_r^{-1} = A_r^{-1} + (A_r^{-1} A_{rk}) K^{-1} (A_{rk} A_r^{-1}) \qquad (10.2\text{ -32})$$

   This completes the global iteration.

It should be noted that if only the arc parameters are being determined, as is the case for "inner" iterations, the solution vector is $d\vec{a'}_r$ and hence the updated arc parameters are computed by

$$\vec{a}_r^{(n+1)} = \vec{a}r^{(n)} + d\vec{a'}_r \tag{10.2 -33}$$

The common parameter matrix K is carried as a symmetric matrix. It is memory-resident throughout the estimation procedure. Its dimension is set by the number of common parameters being determined and remains constant throughout the procedure.

The arc parameter matrices A_r are are also carried as symmetric matrices. Their dimensions vary from arc to arc according to the number of arc parameters being determined. Only one arc parameter matrix A_r and the corresponding covariance matrix A_{rk} are resident in memory at any given time. These arc parameter matrices are stored on disk during step 2 of the above summary and recovered during step 3.

The a priori covariance matrix V_k is not carried as a full matrix. The correlation coefficients between each coordinate of a given station position are carried. The position coordinates of different stations and the geopotential coefficients are considered to be uncorrelated.

The a priori covariance matrix V_r, is not carried as full matrices. The drag coefficient, radiation pressure coefficient, and each bias are considered to be uncorrelated. The covariance matrix. for the epoch elements is carried as a full matrix.

10.3 DATA EDITING

The data editing procedures for GEODYN have two forms:

- hand editing using input cards to delete specific points or sets of points, and

- automatic editing depending on the weighted residual being compared to a given rejection level.

The hand editing is a simple matching of the appropriate GEODYN control card information with the set of observations. The automatic editing of bad observations from a set of data during a data reduction run is performed in GEODYN. Observations are rejected when

$$|\frac{O - C}{\sigma}| > k \tag{10.3 -1}$$

where

O is the observation

C is the computed observation

σ is the a priori standard deviation associated with the observation (input)

k is the rejection level.

The rejection level can apply either for all observations of a given type or for all observations of a given type from a particular station. This rejection level is computed from

$$k = E_M \cdot E_R \tag{10.3 -2}$$

where

E_M is an input multiplier, and

E_R is the weighted RMS of the previous "outer" or global iteration. The initial value of E_R is set on input.

It should be noted that both E_M and E_R have default values.

10.4 ELECTRONIC BIAS

For certain types of electronic tracking data (e.g., Doppler data), biases exist which are different from one pass to the next. In many cases, these biases are of no interest per se, although their existence must be appropriately accounted for if the data is to be used in an orbit or geodetic parameter estimation. In addition, a single data reduction may have hundreds of passes of such electronic data, and the complete solution for each bias would require the use of an excessively large amount of computer memory for storing the normal matrix for the complete set of adjusted parameters.

The effects of electronic biases can be removed, with the use of only a small amount of additional memory, based on the partitioning of the biases from the other parameters being adjusted in the Bayesian least squares estimation. The form which this partitioning takes can be seen from the solution of the basic measurement equation

$$\delta m = B_\epsilon \Delta b + B \Delta X + \epsilon \tag{10.4 -1}$$

where

δm = the vector of residuals $(O - C)$,

Δb = the set of corrections that should be made to the electronic biases,

B_ϵ = the matrix of partial derivatives of the measurements with respect to the biases. The elements of this matrix are either 1's or 0's.

ΔX = the set of corrections to be made to all the other adjustable parameters,

B = the matrix of partial derivatives of the measurements with respect to the X parameters,

ϵ = the measurement noise vector.

The least squares solution of (10.4-1) is

$$\begin{bmatrix} \Delta\hat{b} \\ \Delta\hat{x} \end{bmatrix} = \begin{bmatrix} B_\epsilon^T W B_\epsilon & B_\epsilon^T W B \\ B^T W B_\epsilon & B^T W B \end{bmatrix}^{-1} \begin{bmatrix} B_\epsilon^T W \delta m \\ B^T N \delta m \end{bmatrix} \tag{10.4 -2}$$

with W the weight matrix ($W^{-1} = E(\epsilon\epsilon^T)$), taken to be completely diagonal in GEODYN. The $\Delta\hat{x}$ part of (10.4-2) can be shown to be

$$\begin{aligned} \Delta\hat{x} = {} & [B^T W B - B^T W B_\epsilon (B_\epsilon^T W B_\epsilon)^{-1} B_\epsilon^T W B]^{-1} \\ & \times [B^T W \delta m - B^T W B_\epsilon (B_\epsilon^T W B_\epsilon)^{-1} B_\epsilon^T W \delta m] \end{aligned} \tag{10.4 -3}$$

To effectively remove the electronic bias effects, Eqn. (10.4-3) states that the normal matrix $B^T W B$ must have $B^T W B \epsilon (B_\epsilon^T W B)^{-1} B_\epsilon^T W B$ subtracted from it and the vector $B^T W \delta m$ must have $B^T W B_\epsilon (B_\epsilon^T W B)^{-1} B_\epsilon^T W \delta m$ subtracted from it. Due to the assumed independence of different measurements, it follows that these quantities which must be subtracted are sums of contributions for different passes,

$$B^T W B_\epsilon (B_\epsilon^T W B_\epsilon)^{-1} B_\epsilon^T W B = \sum_{p=1}^{n_b} B_p^T W_p B_{\epsilon p} (B_{\epsilon_p}^T W_p B_{\epsilon_p})^{-1} B_{\epsilon_p}^T W_p B_p \tag{10.4 -4}$$

$$B^T W B_\epsilon (B_\epsilon^T W B_\epsilon)^{-1} B_\epsilon^T W \delta m = \sum_{p=1}^{n_b} B_p^T W_p B_{\epsilon p} (B_{\epsilon_p}^T W_p B_{\epsilon_p}) B_{\epsilon_p}^T W_p \delta m \tag{10.4 -5}$$

where n_b is the total number of passes with electronic biases and the subscript p denotes an array for measurements of pass p. The computation of the right hand sides of (10.4-4) and (10.4-5) requires the arrays

$$\begin{aligned} B_p^T W_p B_p &= na \times 1 \quad \text{array} \\ B_{\epsilon_p}^T W_p B_{\epsilon_p} &= 1 \times 1 \quad \text{array} \\ B_{\epsilon_p}^T W_p \delta m_p &= 1 \times 1 \quad \text{array} \end{aligned} \tag{10.4 -6}$$

where na is the number of adjusted parameters other than biases affecting the arc in which the biases occur. Thus, $na + 2$ storage locations must be assigned for every bias which exists at anyone time.

The individual biases may be adjusted, based on the previous iteration orbital elements and force model parameters. This bias can then be used, along 'with the above accumulated arrays to properly correct the sum of weighted squared residuals upon which the program does dynamic editing. Otherwise, however, it will not be possible for the statistical summaries to incorporate the adjusted values of the electronic biases unless substantial additional core is allocated.

10.5 CONDITION NUMBERS

Condition numbers are the products of the diagonal elements of the matrix A^TWA times the diagonal elements of the matrix $(A^TWA)^{-1}$.

As part of statistical estimation, GEODYN II computes and prints the condition numbers. They provide a measure for the stability of the solution. The closer the condition numbers are to 1, the less ill-conditioned the matrix A^TWA is.

In GEODYN II, subroutine ESTIMA calls subroutine CONDNO to compute the condition numbers.

11 GENERAL INPUT I OUTPUT DISCUSSION

GEODYN is a powerful yet flexible tool for investigating the problems of satellite geodesy and orbit analysis. This same power and flexibility causes extreme variation in both input and output requirements. Consequently, GEODYN contains a great deal of programming associated with input and output.

11.1 INPUT

There are two major functions associated with the input structure. These are the input of:

- Observation data, and

- GEODYN Input Cards.

The observation data utilized by GEODYN includes data from all the major satellite tracking networks. The observational types used to date, together with their originating networks and instrument types are:

- Right Ascension and Declination

SAO Baker- Nunn cameras

USAF PC-1000 cameras

USC GS BC-4 cameras

SPEOPT All of above except Baker-Nunn cameras

- Range

STADAN GRARR S-Band

SAO GSFC Laser

Laser

AMS SECOR

C-Band FPQ-6 Radar

FPS-16 Radar

MSFN S-Band Radar

- Range Rate

STADAN GRARR S-Band

MSFN S- Band Radar

- Frequency Shift

TRANET - Doppler

- Direction Cosines

STADAN Minitrack interferometer

- X and Y Angles

STADAN GRARR

MSFN S-Band Radars

- Azimuth and Elevation Angles

STADAN	GSFC Laser
C-BAND	FPQ-6 Radar
	FPS-16 Radar

- Radar Altimetry

The observations are required to be any of a variety of formats as described in the operations description. The GEODYN Control Cards are the complete specifications for the problem to be solved including special output requests. Their input consists of data and perhaps variances for

- Cartesian orbital elements
- Satellite drag coefficients
- Satellite emissivity
- Zero set measurement biases to be adjusted
- Station positions
- Geopotential and coefficients
- Earth tidal parameters

and data for

- Satellite cross-sectional area
- Satellite mass
- Integration times for the orbit
- Epoch time of elements
- Criteria for iteration convergence and data editing
- Solar and geomagnetic flux

11.2 OUTPUT

The printed output consists of a measurement and residual printout, residual summaries, and solution summaries as detailed below.

For each arc

Measurement and Residual Printout

- Measurement date
- Measurement station
- Measurement type
- Measurement value
- Measurement residual
- Ratio of residual to sigma
- Satellite elevation

Residual Summary by Station and Type

- Station
- Measurement type
- Number of measurements
- Mean of residuals
- Residual RMS about zero
- Number of weighted residuals
- Mean ratio to sigma for weighted residuals
- Randomness measure for weighted residuals
- RMS about zero for weighted residuals

Residual Summary by Type

- Measurement type
- Number of weighted residuals
- Weighted RMS about zero
- Weighted RMS about zero for all types together

Element Summary

- Previous iteration's Cartesian elements
- Adjusted Cartesian elements
- Adjustment to Cartesian elements (delta)
- Standard deviations of fit (sigmas)

- Position RMS
- Velocity RMS
- A priori Kepler elements
- Previous iteration's Kepler elements
- Adjusted Kepler elements
- Adjustment to Kepler elements (delta)
- Double precision adjusted Cartesian elements (current best elements for arc)

Adjusted Force Model Parameter Summary for Arc

- Drag Coefficients, Solar Radiation Pressure Coefficient, and/or resonant geopotential coefficients
- **A priori** coefficient value
- Adjusted coefficient value
- **A priori** standard deviations for coefficient
- Standard deviation of fit for coefficient

Adjusted Parameter Summary

- Instrument biases - timing bias and/or constant bias
- **A priori** bias value
- Adjusted bias value
- **A priori** standard deviation for bias
- Standard deviation of fit for bias
- Time period of coverage

The following items are printed on the last inner iteration of every global iteration:

- Apogee and perigee heights
- Node rate and perigee rate
- Period of the orbit
- Drag rate and period decrement if drag is being applied
- Updated covariance matrix for Cartesian arc elements
- Adjusted arc parameter correlation coefficients

After all arcs

- Total number of weighted measurements for each measurement type
- Total weighted RMS for each measurement type
- Total number of weighted measurements
- Total weighted RMS

Station Summary

- Earth-fixed rectangular coordinates and geodetic (ϕ, λ, h) coordinates
- **A priori** coordinate values
- **A priori** standard deviations for coordinate values
- Adjusted coordinate values
- Standard deviation of fit for coordinate values
- Correlations between determined coordinate values

Geopotential Summary

- C_{nm} and S_{nm} coefficients for each n, m set determined
- **A priori** values
- Adjusted values
- Standard deviations of fit for coefficients

Arc Summary for Global Iteration - For Each Arc

- Updated Cartesian elements for arc
- Correlation coefficients between individual arc parameters
- Standard deviation of fit for arc parameters
- Correlation coefficients between individual arc parameters and parameters common to all arcs

Common Parameter Correlation Coefficients

- Geopotential coefficients
- Cartesian station positions

GEODYN also produces Cartesian ephemeris and a Ground Track listing upon request: In addition an osculating Keplerian element printout is provided on option.

The output from GEODYN consists of the following files which can be provided upon users request:

- Trajectory File: Gives a complete description of the trajectory. Detailed description of the Trajectory File is given in Vol. 4, Section 4.1.
- Residual File: Provides global information, tracking station coordinates, arc information, location data for stations and satellite and residual records for each block of data (see Vol. 4, Section 4.2).
- Partial Derivative File: Outputs measurement partial derivatives (see Vol. 4, Section 4.3).
- E-Matrix File: Outputs the normal equations (see Vol. 4, Section 4.4).
- V-Matrix File: Outputs the force model partial derivatives (see Vol. 4, Section 4.5).

11.3 COMPUTATIONS FOR RESIDUAL SUMMARY

The residual summary information is computed for printing by the program. The formulas used for computing each statistic are presented below.

The mean is

$$\mu_c = \frac{1}{n}\left[\sum_{i=1}^{n} R_i - \sum_{j=1}^{n_b} N_{b_j} b_{e_j}\right] \tag{11.3 -1}$$

where

R_i are the residuals

n is the number of residuals

n_b is the number of electronic biases

N_{b_j} is the number of residuals contributing to the bias computation for b_{ej}

b_{e_j} is the value of the electronic bias.

The RMS is the square root of the sample variance:

$$RMS = \sqrt{s^2} \tag{11.3 -2}$$

where

$$s^2 = \frac{1}{n}\left[\sum_{i=1}^{n} R_i^2 - \sum_{j=1}^{n_b} N_{b_j} b_{e_j}^2\right]$$

The expected value of the sample variance differs from the population variance σ^2:

$$E(s^2) = \sigma^2 - var(\mu_c) \tag{11.3 -3}$$

or rather

$$E(s^2) = \sigma^2(1 - \frac{1}{n}) \tag{11.3 -4}$$

Hence we may make a better estimate of σ^2 by computing

$$\sigma^2 = \frac{n}{n-1}s^2 \tag{11.3 -5}$$

This is known as Bessel's correction. This computed value for the standard deviation, σ is also called the RMS about zero.

The randomness measure used in GEODYN is from a mean square successive difference test. We have

$$RND = \frac{d^2}{s^2} \tag{11.3 -6}$$

when

RND is the random normal deviate, our statistic;

s^2 is the unbiased sample variance; and

$$d^2 = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(R_{i+1} - R_i)^2$$

Note that d^2 is the mean square successive difference. For each i the difference $R_{i+1} - R_i$ has mean zero and variance $2\sigma^2$ under the null hypothesis that $(R_1, ...R_n)$ is a random sample from a population with variance σ^2. The expected value of d^2 is then σ^2. If a trend is present d^2 is not altered nearly so much as the variance estimate s^2, which increases greatly. Thus the critical region RND constant is employed in testing against the alternative of a trend [11 - 1]. In order to use this test, of course, it is necessary to know the distribution of the RND. It can be shown that in the case of a normal population the expected value is given by

$$E(RND) = 1 \qquad (11.3\text{ -7})$$

the variance is given by

$$var(RND) = \frac{1}{n+1}\left[1 - \frac{1}{n-1}\right] \qquad (11.3\text{ -8})$$

and that the test statistic, RND, is approximately normal for large samples ($n > 20$).

11.4 KEPLER ELEMENTS AND NON-SINGULAR KEPLER ELEMENTS

The Kepler elements describe the position of the satellite as referred to an ellipse inclined to the orbit plane. This is shown in Figures 11.4-1 and 11.4-2. The definitions of these elements are:

- a semi-major axis of the orbit
- e eccentricity of the orbit
- i inclination of the orbit plane
- Ω longitude of the ascending node
- ω argument of perigee
- M mean anomaly
- E eccentric anomaly
- f true anomaly

Apogee height and perigee height are sometimes used in place of a and e to describe the shape of the orbit. As can be seen in Figure 11-4-1, the radius at perigee is $a(1 - e)$ and and that at apogee is $a(1 + e)$. The heights are determined by subtracting the radius of the reference ellipsoid at the given latitude from the spheroid height of the satellite. The computations of these last are detailed in Section 5.1.

Conversion to Kepler Elements

The computation of Kepler elements from the Cartesian positions and velocities $x, y, z, \dot{x}, \dot{y}, \dot{z}$ is as follows:

Compute the angular momentum vector per unit mass:

$$\bar{h} = \bar{r} \times \dot{\bar{r}} \tag{11.4 -1}$$

where $\bar{r}$ is the position vector and $\dot{\bar{r}}$ is the velocity vector. Note that $v^2 = \dot{\bar{r}} \cdot \dot{\bar{r}}$. The inclination is given by

$$i = \cos^{-1}\left[\frac{h_z}{h}\right] \tag{11.4 -2}$$

From the vis-viva or energy integral we have

$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right] \tag{11.4 -3}$$

where G is the universal gravitational constant and M is the mass of the primary about which the satellite is orbiting. Thus we have

$$a = \left[\frac{2}{r} - \frac{v^2}{GM}\right]^{-1} \tag{11.4 -4}$$

Recalling Kepler's Third Law,

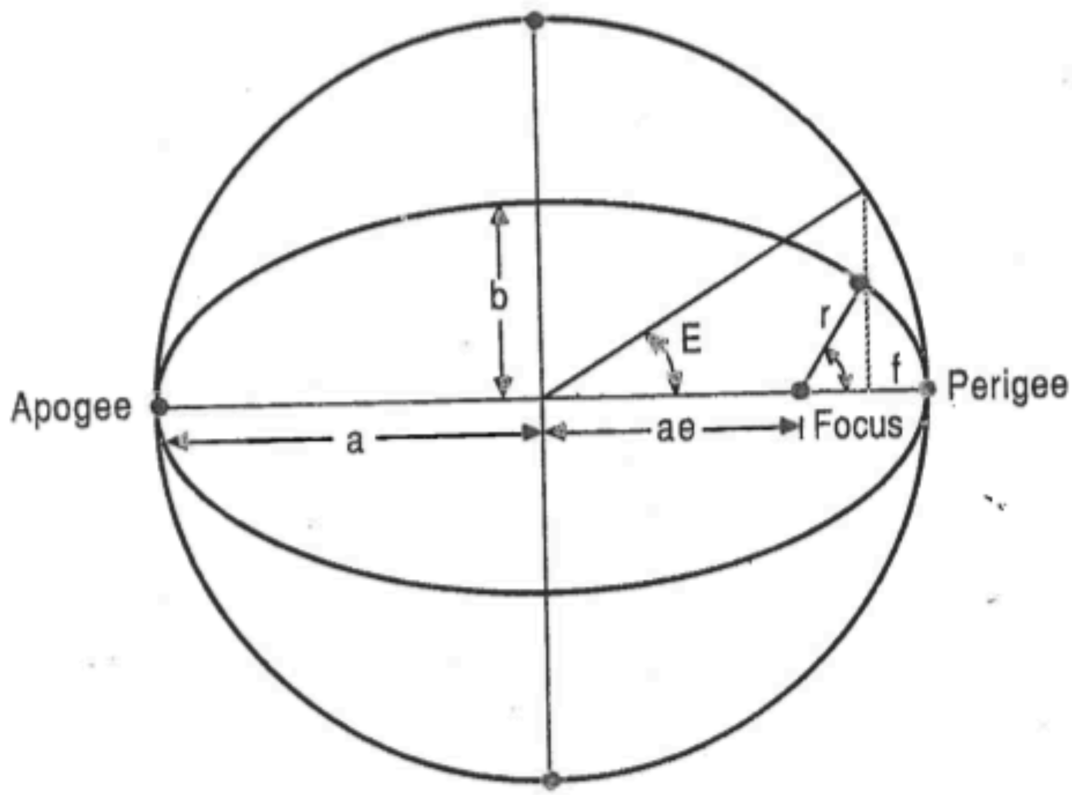
$$h^2 = GMa(1-e)^2 \tag{11.4 -5}$$

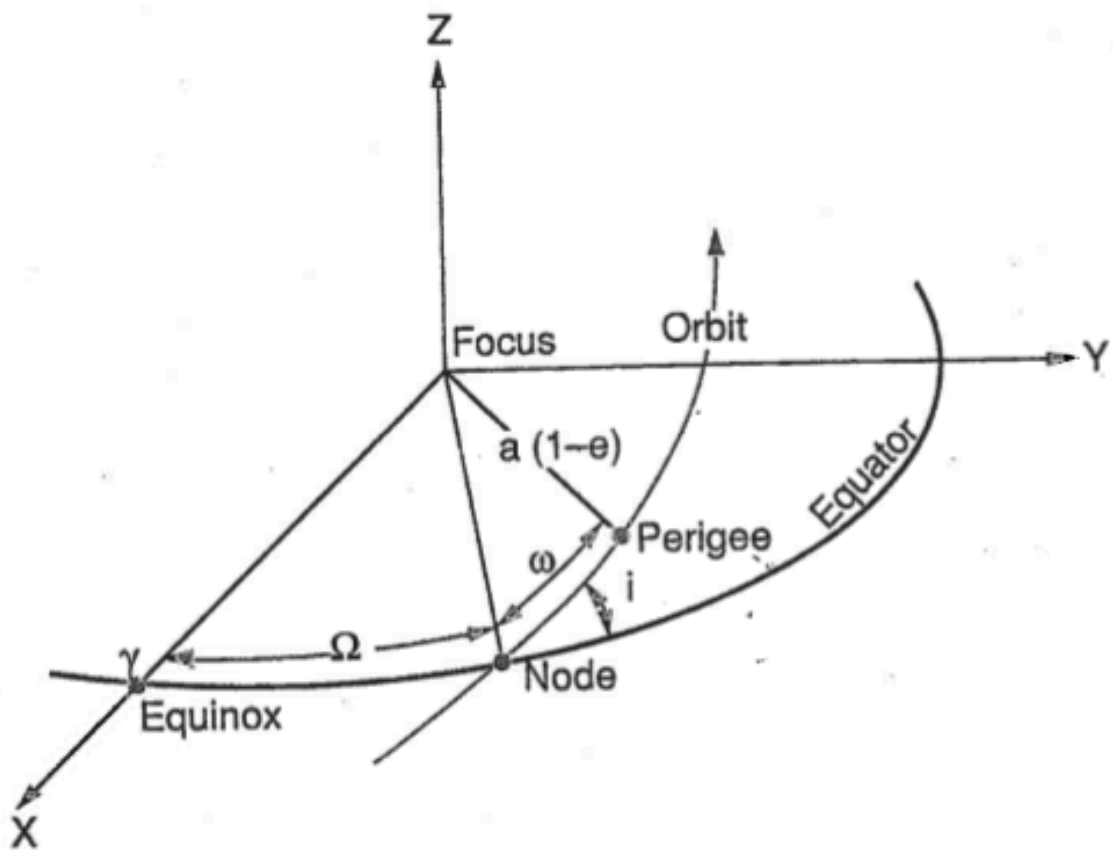
we determine

$$e = \left[1 - \left[\frac{h^2}{aGM}\right]\right]^{\frac{1}{2}} \tag{11.4 -6}$$

The longitude of the ascending node is also determined from the angular momentum vector:

$$\Omega = \tan^{-1}\left[\frac{h_x}{-h_y}\right] \tag{11.4 -7}$$


b
E
r
f
Apogee
Perigee
a
ae
Focus



Z
Focus
Orbit
Y
a (1–e)
Equator
Perigee
ω
i
γ
Ω
Node
Equinox
X

The true anomaly, f, is computed next. Note that in integrating

$$\ddot{\bar{r}} \times \bar{h} = GM\dot{\bar{r}}|\dot{r} \tag{11.4 -8}$$

one arrives at

$$\dot{\bar{r}} \times \bar{h} = GM(\bar{r} + \bar{e}) \tag{11.4 -9}$$

where $\bar{e}$ is a constant of integration of magnitude equal to the eccentricity and pointing toward perihelion. Thus,

$$\bar{r} \times \bar{e} = re \sin f \left[\frac{-\bar{h}}{h} \right] \tag{11.4 -10}$$

or, performing a little algebra,

$$\sin f = \frac{a(1-e^2)\bar{r} \cdot \dot{\bar{r}}}{reh} \tag{11.4 -11}$$

The cosine of the true anomaly comes from

$$r = \frac{a(1-e^2)}{1 + e\cos f} \tag{11.4 -12}$$

that is

$$\cos f = \frac{a(1-e^2)}{er} - \frac{1}{3} \tag{11.4 -13}$$

The true anomaly is then

$$f = \tan^{-1} \left[\frac{\sin f}{\cos f} \right] \tag{11.4 -14}$$

At this point a decision must be made as to whether the orbit is an ellipse ($1 > e > 0$) or a hyperbola ($1 < e < \infty$). For an elliptic orbit, the eccentric anomaly is computed from the true anomaly:

$$\cos E = \frac{\cos f + e}{1 + e\cos f} \tag{11.4 -15}$$

$$\sin E = \frac{\sqrt{1 - e^2 \sin f}}{1 + e\cos f} \tag{11.4 -16}$$

and

$$E = \tan^{-1} \frac{\sin E}{\cos E} \tag{11.4 -17}$$

The mean anomaly is then computed from Kepler's equation:

$$M = E - e \sin E \tag{11.4 -18a}$$

In the case of a hyperbolic orbit, we use an equation analogous to Kepler's equation by introducing F, in place of $E.F$ is defined in terms of f and e using hyperbolic functions;

$$\sinh F = \frac{\sqrt{|1 - e^2|} \sin f}{1 + e \cos f}$$

$$\cosh F = \frac{\cos f + e}{1 + e \cos f}$$

The mean anomaly is

$$M = e \sinh F - F \tag{11.4 -18b}$$

where $F = \ln[\sinh F + \cosh F]$

F is computed by using the definition of sinh and cosh

The central angle u is the angle between the satellite vector and a vector pointing toward the ascending node:

$$\cos u = \frac{x \cos \Omega + y \sin \Omega}{r} \tag{11.4 -19}$$

$$\sin u = \frac{(y \cos \Omega - x \sin \Omega) \cos i + z \sin i}{r} \tag{11.4 -20}$$

$$u = \tan^{-1} \left[\frac{\sin u}{\cos u} \right] \tag{11.4 -21}$$

The argument of perigee is then

$$\omega = u - f$$

Conversion to Non-Singular Kepler Elements

Non-singular Kepler elements consist of
a,
$e\cos(\omega+\Omega)$
$e\sin(\omega+\Omega)$
$\sin\frac{i}{2}\sin\Omega$
$\sin\frac{i}{2}\cos\Omega$
$\Omega+\omega+M$

and are used with low $e(e < 0.005)$ and low $i(i < 1^\circ)$.

The computation of non-singular Kepler elements can easily be performed by using the Kepler elements converted from Cartesian elements, $X, Y, Z, \dot{X}, \dot{Y}$, and $\dot{Z}$.

11.4.1 Partial Derivatives of the Elements

The partial derivatives (P.D.) of the Kepler elements with respect to $x, y, z, \dot{x}, \dot{y}, \dot{z}$ are as follows:

P.D. of the semi-major axis:

$$\frac{\partial a}{\partial s} = +a^2\left[\frac{2}{r^2}\frac{\partial r}{\partial s} + \frac{v}{GM}\frac{\partial v}{\partial s}\right] \tag{11.4 -22}$$

where

$$r = \sqrt{x^2+y^2+z^2}$$
$$v = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

and
$s = x, y, z, \dot{x}, \dot{y}$, and $\dot{z}$, respectively.

That is

$$\frac{\partial a}{\partial s} = \begin{cases} \frac{2a^2}{r^2}\frac{\partial r}{\partial s} & s = x, y, z \\ \frac{2a^2}{GM}v\frac{\partial v}{\partial s} & s = \dot{x}, \dot{y}, \dot{z} \end{cases} \tag{11.4 -23}$$

P.D. of the eccentricity:

$$\frac{\partial e}{\partial s} = \frac{1-e^2}{2ae}\frac{\partial a}{\partial s} - \frac{1-e^2}{he}\frac{\partial h}{\partial s} \tag{11.4 -24}$$

where

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2}$$

and

$$\frac{\partial h}{\partial s} = \frac{1}{h}\left[h_x\frac{\partial h_x}{\partial s} + h_y\frac{\partial h_y}{\partial s} + h_z\frac{\partial h_z}{\partial s}\right] \tag{11.4 -25}$$

See Equations (11-4.26) and (11.4-27). Here:

$$\frac{\partial h}{\partial x} = \dot{y}H_z - \dot{z}H_y$$
$$\frac{\partial h}{\partial y} = \dot{z}H_x - \dot{x}H_z$$
$$\frac{\partial h}{\partial z} = \dot{x}H_y - \dot{y}H_x$$
$$\frac{\partial h}{\partial \dot{x}} = zH_y - yH_z$$
$$\frac{\partial h}{\partial \dot{y}} = xH_z - zH_x$$
$$\frac{\partial h}{\partial \dot{z}} = yH_x - xH_y$$
$$\frac{\partial h_x}{\partial x} = \frac{\partial h_y}{\partial y} = \frac{\partial h_z}{\partial z} = \frac{\partial h_z}{\partial \dot{x}} = \frac{\partial h_y}{\partial \dot{y}} = \frac{\partial h_z}{\partial \dot{z}} = 0 \tag{11.4 -26}$$
$$\frac{\partial h_x}{\partial y} = \dot{z}, \frac{\partial h_x}{\partial z} = -\dot{y}$$
$$\frac{\partial h_x}{\partial \dot{y}} = -z, \frac{\partial h_x}{\partial \dot{z}} = y$$
$$\frac{\partial h_y}{\partial x} = -\dot{z}, \frac{\partial h_y}{\partial z} = \dot{x}$$
$$\frac{\partial h_y}{\partial \dot{x}} = z, \frac{\partial h_y}{\partial \dot{z}} = -x$$
$$\frac{\partial h_z}{\partial x} = \dot{y}, \frac{\partial h_z}{\partial y} = -\dot{x}$$
$$\frac{\partial h_z}{\partial \dot{x}} = -y, \frac{\partial h_z}{\partial \dot{y}} = x$$

and

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial v}{\partial \dot{x}} = \frac{\dot{x}}{v}, \frac{\partial r}{\partial \dot{y}} = \frac{\dot{y}}{v}, \frac{\partial r}{\partial \dot{z}} = \frac{\dot{z}}{v} \tag{11.4 -27}$$

P.D. of the inclination:

$$\frac{\partial i}{\partial s} = \frac{1}{h \sin i}\left[\cos i \frac{\partial h}{\partial s} - \frac{\partial h_z}{\partial s}\right] \tag{11.4 -28}$$

P.D. of the ascending node:

$$\frac{\partial \Omega}{\partial s} = \frac{1}{h_x^2 + h_y^2}\left[h_x \frac{\partial h_y}{\partial s} - h_y \frac{\partial h_x}{\partial s}\right] \tag{11.4 -29}$$

P.D. of the argument of perigee:

$$\frac{\partial \omega}{\partial s} = \frac{\partial u}{\partial s} - \frac{\partial f}{\partial s} \tag{11.4 -30}$$

where

$$\begin{aligned}\frac{\partial u}{\partial s} = \frac{1}{rh^2 \sin i}[zh\cos u + \sin u(xh_y - yh_x)]\frac{\partial h}{\partial s} + [-yh\sin u - xh_z\cos u]\frac{\partial h_x}{\partial s} \\ +[xh\sin u - yh_z\cos u]\frac{\partial h_y}{\partial s} + [-(\bar{r}\cdot\bar{h})\sin u - zh_z\cos u]\frac{\partial h_z}{\partial s} \\ +[hh_y\sin u - h_x h_z\cos u]\frac{\partial x}{\partial s} + [-hh_x\sin u - h_y h_z\cos u]\frac{\partial y}{\partial s} \\ +[h^2\cos u - h^2\cos u]\frac{\partial z}{\partial s}\end{aligned} \tag{11.4 -31}$$

and

$$\begin{aligned}\frac{\partial f}{\partial s} = \frac{1-e^2}{re}\left[\frac{\cos f}{h}(\bar{r}\cdot\dot{\bar{r}}) - \sin f\right]\frac{\partial a}{\partial s} - 2\frac{a}{r}\left[\frac{\cos f}{h}(\bar{r}\cdot\dot{\bar{r}}) - \sin f\right]\frac{\partial e}{\partial s} \\ +\frac{a(1-e^2)}{reh}\left[\frac{\cos f}{h}(\bar{r}\cdot\dot{\bar{r}})\right]\frac{\partial h}{\partial s} + \frac{a(1-e^2)}{reh}\left[\cos f\right]\frac{\partial}{\partial s}(\bar{r}\cdot\dot{\bar{r}}) \\ +\frac{\sin f}{re}\frac{\partial r}{\partial s}\end{aligned} \tag{11.4 -32}$$

P.D. of the mean anomaly:

$$\begin{aligned}\frac{\partial M}{\partial s} &= (1-e\cos E)\frac{\partial E}{\partial s} - \sin E\frac{\partial e}{\partial s} \\ &= \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos f)^2}\left[\frac{\partial f}{\partial s} - \frac{\sin f}{1-e^2}\frac{\partial e}{\partial s}\right] - \sin E\frac{\partial e}{\partial s} \\ &= \frac{r^2}{a^2\sqrt{1-e^2}}\frac{\partial f}{\partial s} - \left[\frac{r}{a(1-e^2)}+1\right]\sin E\frac{\partial e}{\partial s}\end{aligned} \quad (11.4\text{ -33})$$

where

$$\frac{\partial E}{\partial s} = \frac{\sqrt{1-e^2}}{1+e\cos f}\left[\frac{\partial f}{\partial s} - \frac{\sin f}{1-e^2}\frac{\partial e}{\partial s}\right] \quad (11.4\text{ -34})$$

For the case of a hyperbolic orbit,

$$\frac{\partial M}{\partial s} = \frac{r^2}{a^2\sqrt{|1-e^2|}}\frac{\partial f}{\partial s} + \left[\frac{r}{a(1-e^2)}+1\right]\sinh E\frac{\partial e}{\partial s} \quad (11.4\text{ -35})$$

The Partial Derivatives of the Non-Singular Kepler Elements

The partial derivatives (P.D.) of the non-singular Kepler elements with respect to $x, yz, \dot{x}, \dot{y}, \dot{z}$ will be calculated using the partials of the Kepler elements:

$$\frac{\partial a}{\partial s} = \begin{cases} \frac{a^2}{r^2}\frac{\partial r}{\partial s} & s = x, y, z \\ \frac{a^2}{GM}v\frac{\partial v}{\partial s} & s = \dot{x}, \dot{y}, \dot{z} \end{cases} \quad (11.4\text{ -36})$$

$$\frac{\partial}{\partial s}[e\cos(\omega+\Omega)] = \frac{\partial e}{\partial s}\cos(\omega+\Omega) - e\sin(\omega+\Omega)\left[\frac{\partial \omega}{\partial s} + \frac{\partial \Omega}{\partial s}\right]$$

$$\frac{\partial}{\partial s}[e\sin(\omega+\Omega)] = \frac{\partial e}{\partial s}\sin(\omega+\Omega) + e\cos(\omega+\Omega)\left[\frac{\partial \omega}{\partial s} + \frac{\partial \Omega}{\partial s}\right]$$

$$\frac{\partial}{\partial s}\left[\sin\frac{i}{2}\sin\Omega\right] = \frac{1}{2}\cos i\sin\Omega\frac{\partial i}{\partial s} + \cos\Omega\sin\frac{i}{2}\frac{\partial \Omega}{\partial s}$$

$$\frac{\partial}{\partial s}\left[\sin\frac{i}{2}\cos\Omega\right] = \frac{1}{2}\cos i\cos\Omega\frac{\partial i}{\partial s} - \sin\Omega\sin\frac{i}{2}\frac{\partial \Omega}{\partial s}$$

and

$$\frac{\partial}{\partial s}[M+\omega+\Omega]=\frac{\partial M}{\partial s}+\frac{\partial \omega}{\partial s}+\frac{\partial \Omega}{\partial s}$$

where
$S = x, y, z, \dot{x}, \dot{y}$, and $\dot{z}, r = \sqrt{x^2+y^2+x^2}$ and $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$

Conversion to an Alternate Set of Kepler Elements and Their Partial Derivatives

An alternate set of Kepler elements consists of

$$a, e, i, \Omega, \omega+M, M \tag{11.4 -37}$$

These elements are the same as the previous Kepler elements except $\omega+M$. This can be calculated by simply adding ω and M that have been calculated in Kepler elements.

Their partial derivatives with respect to $x, y, z, \dot{x}, \dot{y}$, and $\dot{z}$ should be the same as the partials of Kepler elements previously calculated. The only difference is the partials of $\omega+M$. Thus partial derivatives can be calculated by adding the individual partial derivates ω and M.

Conversion from Kepler Elements The input elements are considered to be a, e, i, Ω, ω, and M and the Cartesian elements are required. An iterative procedure, Newton's method, is used to recover the eccentric anomaly. For an elliptic orbit, the iterative procedure uses Kepler's equation ($M = E - e\sin E$), to successively compute

$$E' = E = \frac{E - e\sin E - M}{1 - e\cos E}$$

For a hyperbolic orbit, the iterative procedure uses

$$F' = F - \frac{e\sinh F - F - M}{e\cosh F - 1}$$

where $F, \sinh F$ and $\cosh F$ are defined previously.

This procedure for converting $a, e, i, \Omega, \omega, M$ to $x, y, z, \dot{x}, \dot{y}, \dot{z}$ is performed in the GEODYN system by subroutine POSVEL.

The vectors $\bar{A}$ and $\bar{B}$ are computed. $\bar{A}$ is a vector in the orbit plane directed toward pericenter with a magnitude equal to the semi-major axis of the orbit:

$$\bar{A} = a \begin{bmatrix} \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\ \cos\omega\sin\Omega + \sin\omega\sin\Omega\cos i \\ \sin\omega\sin i \end{bmatrix} \tag{11.4 -38}$$

$\bar{B}$ is a vector in the orbit plane directed 90° counterclockwise from $\bar{A}$ with a magnitude equal o the semi-minor axis of the orbit:

$$\bar{B} = a\sqrt{1-e^2} \begin{bmatrix} -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos\omega\sin i \end{bmatrix} \tag{11.4 -39}$$

The position vector $\bar{r}$ is then

$$\bar{r} = (\cos E - e)\bar{A} + (\sin E)\bar{B} \tag{11.4 -40}$$

The velocity vector is given by

$$\dot{\bar{r}} = \dot{E}[(-\sin E)\bar{A} + (\cos E)\bar{B}] \tag{11.4 -41}$$

where $\dot{E}$ is given by

$$\dot{E} = \frac{\sqrt{\frac{GM}{a^3}}}{1 - e\cos E} \tag{11.4 -42}$$

11.4.2 Node Rate and Perigee Rate

The node rate $\dot{\Omega}$ and perigee rate $\dot{\omega}$ are computed from Lagrange's Planetary Equations. As these are for printout only, GEODYN uses just the Earth oblateness term ill the geopotential. From Reference [11 - 4] , page 39, we have

$$\dot{\Omega} = \left[\frac{3}{2}C_{20}\sqrt{\frac{GM}{a_e^2}}\right]\left[\frac{a}{a_e}\right]^{-3.5}\frac{\cos i}{(1-e^2)^2} \tag{11.4 -43}$$

$$\dot{\omega} = \left[\frac{3}{4}C_{20}\sqrt{\frac{GM}{a_e^2}}\right]\left[\frac{a}{a_e}\right]^{-3.5}\frac{1-5\cos^2 i}{(1-e^2)^2} \tag{11.4 -44}$$

in radians per second, or

$$\dot{\Omega} = -9.97\left[\frac{a}{a_e}\right]^{-3.5} \frac{\cos i}{(1-e^2)^2} \quad (11.4\text{ -}45)$$

$$\dot{\omega} = -4.98\left[\frac{a}{a_e}\right]^{-3.5} \frac{1 - 5\cos^2 i}{(1-e^2)^2} \quad (11.4\text{ -}46)$$

in degrees per day. The quantities used in the above equations are defined as:

a_e is the semi-major axis of the Earth

GM is the product of the universal gravitational constant G and the mass of the Earth M

C_{20} is the Earth oblateness term ill the geopotential (see Section 8.3)

a semi-major axis of the orbit

e eccentricity of the orbit

i inclination of the orbit

11.4.3 Period Decrement and Drag Rate

The period decrement and the drag rate are determined from the partial derivatives of the position and velocity with respect to the drag coefficient at the final integrator time step in the given arc. These (multiplied by the drag coefficient) . represent the sensitivity of the position or velocity to drag effects. Let us define

$$\bar{\Delta D} = \frac{\partial}{\partial C_D}(\bar{r}) \cdot C_D \quad (11.4\text{ -}47)$$

where

r is the satellite (inertial) position vector

C_D is the drag coefficient

We also define

$$\dot{\bar{\Delta D}} = \frac{\partial}{\partial C_D}(\dot{\bar{r}}) \cdot C_D \quad (11.4\text{ -}48)$$

The (two-body) period of the orbit is

$$P = 2\pi\sqrt{\frac{a^3}{GM}} \quad (11.4\text{ -}49)$$

where

a is the semi-major axis of the orbit

GM is the product of the universal gravitational constant G and the mass of the Earth M

Thus,

$$\Delta P = 3\pi\sqrt{\frac{a}{GM}}\Delta a \tag{11.4 -50}$$

The vis-viva or energy integral has

$$v^2 = GM[\frac{2}{r} - \frac{1}{a}] \tag{11.4 -51}$$

hence

$$a = \frac{1}{[\frac{2}{r} - \frac{\dot{\bar{r}}\cdot\dot{\bar{r}}}{GM}]} \tag{11.4 -52}$$

Recognizing that $\Delta(\bar{r})$ is $\Delta\bar{D}$ and $\Delta(\dot{\bar{r}})$ is $\Delta\dot{\bar{D}}$

$$\Delta a = 2a\left[\frac{\dot{\bar{r}}\cdot\Delta\bar{D}}{2r^2} + a\frac{\dot{\bar{r}}\cdot\Delta\dot{\bar{D}}}{GM}\right] \tag{11.4 -53}$$

The effect of the drag on the period is then given by

$$\Delta P = 6\pi\sqrt{\frac{a^3}{GM}}\left[\frac{\dot{\bar{r}}\cdot\Delta\bar{D}}{2r^2} + a\frac{\dot{\bar{r}}\cdot\Delta\dot{\bar{D}}}{GM}\right] \tag{11.4 -54}$$

The daily rate or period decrement is computed as $\Delta P/\Delta t$ where Δt is the elapsed time (in days) between the last integrator time point and epoch.

The drag rate is computed from the along track (actually normal) portion of $\Delta\bar{D}$ that is ΔD_N. We need to construct the unit vector along track, $\hat{L}$. The velocity vector r may be resolved into a radial component and a component normal to the radius vector. The magnitude of the nodal component is found by the Pythagorean Theorem:

$$A = \sqrt{\dot{\bar{r}}\cdot\dot{\bar{r}} - [\frac{1}{r}\bar{r}\cdot\dot{\bar{r}}]^2} \tag{11.4 -55}$$

The unit normal vector $\hat{L}$ is then

$$\hat{L} = [\dot{\bar{r}} - \frac{1}{r}\bar{r}\cdot\dot{\bar{r}}]/A \tag{11.4 -56}$$

The normal portion of $\Delta \bar{D}$ is then

$$\Delta D_N = \hat{L} \cdot \Delta \bar{D} \tag{11.4 -57}$$

This $\Delta \bar{D}_N$ represents the along-track position effect due to drag over the integrated time span. The drag rate is computed as $\Delta D_N / \Delta t^2$ where Δt is again the elapsed time in days.

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